PROOF OF THE DUBROVIN CONJECTURE AND ANALYSIS OF THE TRITRONQUÉE SOLUTIONS OF P_I

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Abstract. We show that the tritronquée solution of the Painlevé equation $\mathrm{P_I}$, $y'' = 6y^2 + z$ which is analytic for large z with $\arg z \in \left(-\frac{3\pi}{5}, \pi\right)$ is pole-free in a region containing the full sector $\left\{z \neq 0, \arg z \in \left[-\frac{3\pi}{5}, \pi\right]\right\}$ and the disk $\left\{z : |z| < \frac{37}{20}\right\}$. This proves in particular the Dubrovin conjecture, an open problem in the theory of Painlevé transcendents. The method, building on a technique developed in [4], is general and constructive. As a byproduct, we obtain the value of the tritronquée and its derivative at zero within less than 1/100 rigorous error bounds.

1. Introduction and Main result

Understanding the global behavior in $\mathbb{C}$ of the tritronquée solutions (see below) of the Painlevé equation $\mathrm{P_I}$ is essential in a number of problems such as the critical behavior in the NLS/Toda lattices ([6], [7]) and the analysis of the cubic oscillator ([13]). Considerations related to the behavior of NLS/Toda solutions corroborated by numerical evidence led to the Dubrovin conjecture, see [6] (cf. also [2] and [7]). This conjecture states that the tritronquée solutions are analytic in a neighborhood of the origin O and in a sector of width $8\pi/5$ containing O. A number of partial results on this question have been obtained so far, see e.g. [8], [12]–[14] but, in spite of the existence of an underlying Riemann-Hilbert representation, at the time of the present paper the conjecture is still open.

The purpose of the present paper is to prove the Dubrovin conjecture alongside other results about the tritronquées.

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We write $\mathrm{P_I}$ in the form

$$y'' = 6y^2 + z \tag{1}$$

Tritronquées. There are exactly five solutions of (1) which are analytic for large z in a sector of width $8\pi/5$. These special solutions called *tritronquées* are obtained from each-other through the five-fold symmetry of $\mathrm{P_I}$, $y \mapsto e^{4ik\pi/5} y\left(e^{2\pi ik/5} z\right)$, $k = 0, \ldots, 4$ (cf. [9],[11], [8]). To understand their properties it is clearly enough to analyze one of them; we choose the solution y_t uniquely defined by the property $y(z) \sim i\sqrt{z/6}[1 + O(z^{-5/2})]$ as $z \to +\infty$ (cf. *e.g.* Proposition 2 and Theorem 3 in [8]). The main result of this paper is

Theorem 1. *The tritronquée y_t is analytic in the region*

$$\left\{z \neq 0 : \arg\, z \in [-3\pi/5, \pi]\right\} \cup \left\{z : |z| < \tfrac{37}{20}\right\} \tag{2}$$

Corollary 1. *Dubrovin's conjecture, stating that y_t is free of singularities in $\left\{z : \arg z \in \left[-\frac{3}{5}\pi, \pi\right]\right\}$, holds.*

Note 1. (i) The proof of Theorem 1 is organized as follows. We first establish the analyticity of y_t in the region $\{z : |z| > 1.7,\ \arg z \in (-3\pi/5, \pi)\}$, and continuity in its closure. We determine $y_t(z_0), y_t'(z_0),\ z_0 = 1.7e^{i\pi/5}$ with errors smaller than $6.5 \cdot 10^{-3}$. We then show that any solution of $\mathrm{P_I}$ with $y_t(z_0), y_t'(z_0)$ in this range matches a Maclaurin series solution of (1) with a radius of convergence bounded below by $37/20 = 1.85$. The result follows.

(ii) As a byproduct we find within rigorous errors bounds smaller than 1/100 the values of $y(0)$ and $y'(0)$, which are also needed in applications, and these values are consistent with the numerical calculation by Joshi and Kitaev [8].

(ii) The lower bound 1.85 is not optimal, yet not far from the numerically obtained radius of analyticity, ~ 2.38 ([8]). The methods we use can, in principle, be adapted to rigorously calculate the tritronquée and the position of its first pole with any prescribed accuracy.

We will also use a standard normalization of $\mathrm{P_I}$ [1], similar to the Boutroux form. After the change of variables

$$x = \frac{e^{i\pi/4}}{30}\,(24z)^{5/4}\,,\ y(z) = i\sqrt{\frac{z}{6}}\left(1 - \frac{4}{25x^2} + h(x)\right) \tag{3}$$

$\mathrm{P_I}$ becomes

$$h'' + \frac{1}{x}h' - h = \frac{h^2}{2} + \frac{392}{625x^4}. \tag{4}$$

Using the change of variables leading to (4), the aforementioned reference [8] implies that there is a unique solution h of (4) with the behavior $h(x) = O\left(x^{-4}\right)$ as $x \to +i\infty$.

2. The tritronquée y_t on the anti-Stokes line $z = e^{i\pi/5}\mathbb{R}^+$

First, we consider behavior of y_t on the anti-Stokes line $z = e^{i\pi/5}\mathbb{R}^+$ for $|z| \geqslant \frac{1}{24}(30)^{4/5}$ $(= 0.633\cdots)$. We find $y_t(z_0)$ and $y_t'(z_0)$ for $z_0 = 1.7e^{i\pi/5}$, accurately and with rigorous error bounds, later used to show that any solution of $\mathrm{P_I}$ within the proved range has y_t with a power series at the origin with radius of convergence exceeding 1.85. After the substitution

$$h(x) =: \frac{H(x)}{\sqrt{x}}, \tag{5}$$

(4) implies

$$H'' - \left(1 - \frac{1}{4x^2}\right)H = \frac{H^2}{2\sqrt{x}} + \frac{392}{625x^{7/2}} \tag{6}$$

We define

$$\Omega_I := \left\{x \in i\mathbb{R}^+, |x| \geqslant \rho\right\}; \quad \text{where } \rho \geqslant 1 \tag{7}$$

and the Banach space

$$\mathcal{S}_I = \{H \in C(\Omega_I, \mathbb{C}) : \|H\| < \infty\} \tag{8}$$

where $\|\cdot\|$ is the weighted norm

$$\|H\| = \sup_{x \in \Omega_I} \left|x^{5/2}H\right| \tag{9}$$

We invert (6) in $\mathcal{S}_I$ to obtain the integral equation

$$H(x) = H_0(x) + \int_{i\infty}^{x} \sinh(x-t)\left\{-\frac{H(t)}{4t^2} + \frac{H^2(t)}{2\sqrt{t}}\right\} dt =: \mathcal{N}[H](x), \tag{10}$$

where

$$H_0(x) = \int_{i\infty}^{x} \sinh(x-t)\ \frac{392}{625t^{7/2}}dt \tag{11}$$

In the process of inverting (6), no nonzero linear combination $C_1e^x + C_2e^{-x}$ may be added to the right side of (10) as $e^{\pm x} \notin \mathcal{S}_I$. We note that $|x| = \rho = 1$ corresponds to $|z| = \frac{30^{4/5}}{24} = 0.6331\cdots$.

Lemma 2. *There exists a unique solution H to (10) and therefore of (6) in $\mathcal{S}_I$.*

Proof. Using $|\sinh(x-t)| \leqslant 1$ for $x, t \in \Omega_I$ and (11) we obtain

$$|H_0(x)| \leqslant \frac{784}{3125|x|^{5/2}} \Rightarrow \|H_0\| \leqslant \frac{784}{3125} \tag{12}$$

Since $|H(t)| \leqslant |t|^{-5/2}\ \|H\|$, it follows that

$$\left|\int_{i\infty}^{x} \sinh(x-t)\left\{-\frac{H(t)}{4t^2} + \frac{H^2(t)}{2\sqrt{t}}\right\} dt\right| \leqslant \frac{\|H\|}{14|x|^{7/2}} + \frac{\|H\|^2}{9|x|^{9/2}},$$

Therefore, it follows that

$$\|\mathcal{N}[H]\| \leqslant \|H_0\| + \frac{\|H\|}{14\rho} + \frac{\|H\|^2}{9\rho^2},$$

and similarly,

$$\|\mathcal{N}[H_1] - \mathcal{N}[H_2]\| \leqslant \left(\frac{1}{14\rho} + \frac{\|H_1\| + \|H_2\|}{9\rho^2}\right)\|H_1 - H_2\|$$

Consider now the ball $B \subset \mathcal{S}_I$ of radius $(1+\varepsilon)\|H_0\|$ for some $\varepsilon > 0$ to be chosen shortly. For any $H, H_1, H_2 \in B$ we obtain from the inequalities above

$$\|\mathcal{N}[H]\| \leqslant \|H_0\| \left(1 + \frac{1+\varepsilon}{14\rho} + \frac{(1+\varepsilon)^2\|H_0\|}{9\rho^2}\right) \tag{13}$$

$$\|\mathcal{N}[H_1] - \mathcal{N}[H_2]\| \leqslant \left\{\frac{1}{14\rho} + \frac{2}{9\rho^2}(1+\varepsilon)\|H_0\|\right\} \|H_1 - H_2\| \tag{14}$$

The estimates (13) and (14) imply $\mathcal{N} : B \to B$ and is contractive if

$$\frac{1+\varepsilon}{14\rho} + \frac{(1+\varepsilon)^2\|H_0\|}{9\rho^2} \leqslant \varepsilon, \text{and } \frac{1}{14\rho} + \frac{2}{9\rho^2}(1+\varepsilon)\|H_0\| < 1 \tag{15}$$

Recalling that $\rho \geqslant 1$, (12) implies that the conditions in (15) are satisfied if $\varepsilon = \frac{3}{20}$,. Thus, (10) has a unique solution H in a ball of size $\frac{23}{20}\|H_0\|$– and so does the equivalent equation 6). Reverting the changes of variables, the corresponding y is y_t since $H \in \mathcal{S}_I$ implies the decay of y at $i\infty$ that uniquely determines y_t. ∎

Lemma 3. *With H as in Lemma 2, we have for $x \in \Omega_I$,*

$$\left|H'(x)\right| \leqslant \left(\frac{1}{14|x|^{7/2}}\|H\| + \frac{1}{9|x|^{9/2}}\|H\|^2 + \frac{392}{625|x|^{7/2}}\right),$$

Proof. Differentiating (10) and using (11) we obtain

$$H'(x) = \int_{i\infty}^{x} \cosh(x-t)\left\{-\frac{H(t)}{4t^2} + \frac{H^2(t)}{2\sqrt{t}}\right\} dt - \int_{i\infty}^{x} \sinh(x-t)\frac{7\cdot 392}{2\cdot 625 t^{9/2}} dt, \tag{16}$$

where the last term is the result of integration by parts. Since $|H(t)| \leqslant t^{-5/2}\|H\|$, $|\sinh(x-t)|, |\cosh(x-t)| \leqslant 1$, for $x, t \in \Omega_I$ in (16), the result follows. ∎

Remark 2. In order to obtain small error bounds for H and H' at $x = x_0 := i\,(24\cdot 1.7)^{5/4}/30 = i3.437\cdots$ corresponding to $z = z_0 = e^{i\pi/5}1.7$, a good choice is $\rho = |x_0| = 3.437\cdots$ and $\varepsilon = \frac{1}{40}$ (for which the conditions in (15) in Lemma 2 hold). With this choice, $|H(x_0)| \leqslant |x_0|^{-5/2}\|H\| \leqslant \frac{41}{40}|x_0|^{-5/2}\|H_0\| = \frac{41}{40}\frac{784}{3125}|x_0|^{-5/2}$. Also, Lemma 3 is applied to bound $H'(x_0)$ by using $\|H\| \leqslant \frac{(41)(784)}{(40)(3125)}$. By (1), (3) and (5) we have

$$y(z) = i\sqrt{\frac{z}{6}}\left(1 - \frac{4}{25x^2} + \frac{H(x)}{\sqrt{x}}\right), x = e^{i\pi/4}\frac{(24z)^{5/4}}{30}, \tag{17}$$

Defining

$$y_0(z) = i\sqrt{\frac{z}{6}}\left(1 - \frac{4}{25x^2}\right),$$

straightforward calculations show that

$$y(z_0) - y_0(z_0) = i\sqrt{\frac{z_0}{6x_0}}H(x_0),$$

$$y'(z_0) - y_0'(z_0) = i\sqrt{\frac{z_0}{6}}\left(\frac{1}{z_0\sqrt{x_0}}\right)\left(-\frac{H(x_0)}{8} + \frac{5}{4}x_0H'(x_0)\right),$$

and thus, using the obtained bounds on $H(x_0)$ and $H'(x_0)$, we get

$$\left|y(z_0) - y_0(z_0)\right| \leqslant \frac{3}{890}, \ \left|y'(z_0) - y_0'(z_0)\right| \leqslant \frac{29}{4468} \tag{18}$$

Note also that

$$y_0(z_0) = i\sqrt{\frac{z_0}{6}}\left(1 - \frac{4}{25x_0^2}\right) = C_1e^{-2i\pi/5}, \ C_1 = -0.5394994\cdots \tag{19}$$

$$y_0'(z_0) = i\sqrt{\frac{6}{z_0}}\left(\frac{1}{12} + \frac{4}{75x_0^2}\right) = C_2e^{i2\pi/5}, \ C_2 = 0.148075\cdots \tag{20}$$

Remark 3. *The function h is real-valued in Ω_I. Indeed, with $x = i\tau$, (3) becomes*

$$h'' + \frac{1}{\tau}h' + h + \frac{h^2}{2} + \frac{392}{625\tau^4} = 0 \tag{21}$$

By complex conjugation symmetry, if $h(\tau)$ is a solution of (21) then so is $\overline{h(\overline{\tau})}$; by uniqueness, $h(\tau)$ is real-valued.

3. The tritronquée in the region $x \in \Omega_1 \cup \Omega_2$

Consider the domains

$$\Omega_1 := \left\{x: \ |x| \geqslant \rho_0 \ , \ \frac{\pi}{4} \leqslant \arg x \leqslant \frac{\pi}{2}\right\}; \ \Omega_2 := \left\{x: \ -\frac{\pi}{4} \leqslant \arg x \leqslant \frac{\pi}{4} \ , \ \mathrm{Re}\ x \geqslant \frac{\rho_0}{\sqrt{2}}\right\},$$

and define the Banach space $\mathcal{S}_2$ of analytic functions in the interior of $\Omega_1 \cup \Omega_2$, continuous up to the boundary, equipped with the weighted norm

$$\|H\| = \sup_{x\in\Omega_1\cup\Omega_2} \left|x^{5/2}H\right|$$

We consider the operator $\mathcal{N}$, defined in (10), now acting on $\mathcal{S}_2$.

Lemma 4. *For $\rho_0 \geqslant \frac{1}{30}\left(24 \cdot 1.7\right)^{5/4} = |x_0|$, there exists a unique solution to the integral equation (10), $H = \mathcal{N}[H]$, in $\mathcal{S}_2$, corresponding to the tritronquée through the transformation (17).*

Proof. We first obtain bounds on

$$H_0(x) = \int_{i\infty}^{x} \sinh(x-t)\frac{392}{625t^{7/2}} = \int_{i\infty}^{x} e^{x-t}\ \frac{196}{625t^{7/2}}dt - \int_{i\infty}^{x} e^{t-x}\ \frac{196}{625t^{7/2}}dt \tag{22}$$

Consider first $x \in \Omega_2$, The contour in the middle integral in (22) can be deformed to a radial one joining ∞ to $x \in \Omega_1 \cup \Omega_2$ for which $|e^{x-t}| \leqslant 1$ implying

$$\left|x^{5/2}\int_{\infty}^{x} e^{x-t}\frac{1}{t^{7/2}}dt\right| \leqslant \frac{2}{5} \tag{23}$$

For the last integral in (22), no such radial path deformation is possible because of growth of e^t. On the vertical integration path, we paramerize $t = x + i\tau$, $\tau \in \mathbb{R}^+$. If we separate out real and imaginary parts of x: $x = a + ib$, then $x \in \Omega_2$ implies $\frac{|x|}{\sqrt{2}} \leqslant \mathrm{Re}\ x = a$ and $|b| \leqslant a$. Then,

$$\begin{aligned} \Big|\int_{i\infty}^{x} e^{t-x}\ \frac{1}{t^{7/2}}dt\Big| &\leqslant \Big|\int_{0}^{\infty}\left(a^2+(b+\tau)^2\right)^{-7/4}d\tau\Big| \\ &\leqslant \frac{1}{a^{5/2}}\int_{-b/a}^{\infty}\left(1+p^2\right)^{-7/4}dp \leqslant \frac{1}{a^{5/2}}\int_{-1}^{\infty}\left(1+p^2\right)^{-7/4}dp \leqslant \frac{2^{5/4}}{|x|^{5/2}}\int_{-1}^{\infty}\left(1+p^2\right)^{-7/4}dp \end{aligned} \tag{24}$$

Combining with (23) it follows that for $x \in \Omega_2$ we have

$$\left|x^{5/2}H_0(x)\right| \leqslant \frac{196}{625}\left[2^{5/4}\int_{-1}^{\infty}(1+p^2)^{-7/4}dp + \frac{2}{5}\right] \leqslant \frac{32}{25}$$

For $x \in \Omega_1$, we note that $|t| \geqslant |x|$, and on a vertical contour of integration $|\cosh(x-t)| \leqslant 1$ and $|dt| \leqslant \sqrt{2}d|t|$ and therefore

$$\left|H_0(x)\right| \leqslant \sqrt{2}\int_{|x|}^{\infty}\frac{392}{625|t|^{7/2}}d|t| \leqslant \frac{784\sqrt{2}}{3125|x|^{5/2}}$$

This is clearly a smaller bound than the one for $x \in \Omega_2$. Therefore, for any $x \in \Omega_1 \cup \Omega_2$, we have

$$\|H_0\| \leqslant \frac{32}{25} =: M$$

For the nonlinear term, the calculations are similar. For $x \in \Omega_1$, on the vertical path, since $|\sinh(x-t)| \leqslant 1$, $|H(t)| \leqslant |t|^{-5/2}\|H\|$ and $|dt| \leqslant \sqrt{2}d|t|$,

$$\Big|\int_{i\infty}^{x}\sinh(x-t)\frac{H^2(t)}{2\sqrt{t}}dt \leqslant \frac{\sqrt{2}}{9|x|^{9/2}}\|H\|^2$$

For $x \in \Omega_2$, we split sinh into two exponentials and break the integral accordingly; in one of the integrals the contour can be deformed into a radial path. In the other, we parametrize the vertical path as in the estimates of H_0. Since the bound in Ω_1 is clearly smaller, this results in

$$\left\| \int_{i\infty}^{x} \sinh(x-t) \frac{H^2(t)}{2\sqrt{t}} dt \right\| \leqslant N \rho_0^{-2} \|H\|^2, \tag{25}$$

where

$$N = \frac{1}{18} + 2^{1/4} \int_{-1}^{\infty} \left[1+p^2\right]^{-11/4} dp \leqslant \frac{203}{138} \tag{26}$$

Now consider the linear term.

$$-\int_{i\infty}^{x} \sinh(x-t) \frac{H(t)}{4t^2} dt$$

For $x \in \Omega_1$, as before, we obtain using $|\sin(x-t)| \leqslant 1$ on a vertical path, $|dt| \leqslant \sqrt{2} d|t|$ and $|H(t)| \leqslant \|H\| |t|^{-5/2}$,

$$\left| \int_{i\infty}^{x} \sinh(x-t) \frac{H(t)}{4t^2} dt \right| \leqslant \frac{\sqrt{2}}{14 \rho_0 |x|^{5/2}} \|H\|$$

For $x \in \Omega_2$, similarly writing $\sinh(x-t)$ in exponential form, breaking the integral accordingly and separately estimating each term we get

$$\left| \int_{i\infty}^{x} \sinh(x-t) \frac{H(t)}{4t^2} dt \right| \leqslant \frac{L}{\rho_0 |x|^{5/2}} \|H\|,$$

$$L = \frac{1}{28} + 2^{-5/4} \int_{-1}^{\infty} (1+p^2)^{-9/4} dp \leqslant \frac{3}{5}$$

Clearly, the bound for $x \in \Omega_2$ is larger. We conclude that

$$\left\| \int_{i\infty}^{x} \sinh(x-t) \frac{H(t)}{4t^2} dt \right\| \leqslant \frac{L}{\rho_0} \|H\|$$

$\mathcal{N} : B \to B$ is contractive if for some $\varepsilon > 0$ we have

$$L \rho_0^{-1} (1+\varepsilon) + N \rho_0^{-2} M (1+\varepsilon)^2 \leqslant \varepsilon \ , \ L \rho_0^{-1} + 2N \rho_0^{-2} M (1+\varepsilon) < 1 \tag{27}$$

Both conditions are satisfied for $\rho_0 = |x_0|$, when $\varepsilon = \frac{3}{2}$, *i.e.* ball size is $\frac{5}{2}M$, and the lemma follows from contraction mapping theorem, and the fact that Ω_I is the boundary of Ω_1 implies that this solution is the same in Lemma 2, –corresponding to the tritronquée. ∎

4. Analysis of y_t for $x \in \Omega_4$

Definition 5. *Let*

$$\Omega_4 := \left\{ x : \ |x| \geqslant \rho \geqslant 3 \ , \ \arg x \in \left[-\frac{\pi}{2}, -\frac{\pi}{4} \right] \right\}$$

The results in [1] imply that for large x in Ω_4 s.t. $\left| \frac{Se^{-x}}{\sqrt{x}} - 12 \right| \geqslant m > 0$ has the uniform asymptotic expansion

$$h(x) \sim \sum_{j=0}^{\infty} x^{-j} F_j(\xi),$$

where $\xi = \frac{Se^{-x}}{\sqrt{x}}$, and each $F_j(\xi)$ is a rational function with poles at $\xi = 12$. Further, since the Stokes constant $S = i\sqrt{\frac{6}{5\pi}} = i0.61804..$ (cf. [10], [11]) is moderately small, ξ is also moderately small for $x \in \Omega_4$ even for $\rho = 3$. Therefore, a few terms in the Taylor series of each F_j at $\xi = 0$ and a fairly small number of F_j's are expected to yield a good approximation of $h(x)$.

With this expectation, we choose an approximate expression h_0 for h_t in Ω_4 in the form

$$h_0(x) = \left(\xi + \frac{\xi^2}{6} + \frac{\xi^3}{48} + \frac{\xi^4}{432} + \frac{5\xi^5}{20736} \right) + \frac{1}{x} \left(-\frac{\xi}{8} - \frac{11}{72}\xi^2 - \frac{43}{1152}\xi^3 \right) + \frac{9\xi}{128x^2}, \tag{28}$$

where

$$\xi = \frac{Se^{-x}}{\sqrt{x}}, \; S = i\sqrt{\frac{6}{5\pi}} \tag{29}$$

Let

$$h(x) = h_0(x) + x^{-1/2}G(x) \tag{30}$$

By (4), $G(x)$ satisfies

$$G'' - (1 + h_0(x))\, G(x) = \frac{G^2(x)}{2\sqrt{x}} - R(x) - \frac{G(x)}{4x^2}, \tag{31}$$

where $R(x)$ is given by

$$R(x) = \sqrt{x}\left(h_0'' + \frac{1}{x}h_0' - h_0(x) - \frac{1}{2}h_0^2(x) - \frac{392}{625x^4}\right) = \sum_{j=5}^{9} x^{-j/2} r_j\left(e^x\right) \tag{32}$$

and the $r_j(\zeta)$ are polynomials in ζ^{-1}, where only r_7 has a nonzero constant term $-\frac{392}{625}$ (see (143)-(146) in the Appendix for the precise expressions of r_j). Define

$$R_0(x) = x^{-5/2} r_5(e^x)\ ,\ R_1(x) = x^{-3}\ r_6(e^x)\ ,\ \tilde{R}(x) = \sum_{j=7}^{9} x^{-j/2} r_j(e^x) = R - R_0 - R_1 \tag{33}$$

and

$$\tilde{r}_7(\zeta) = r_7(\zeta) + \frac{392}{625}\ ,\ \tilde{r}_j(\zeta) = r_j(\zeta) \text{ for } j \neq 7 \tag{34}$$

To write (31) in integral form, we need the properties of the Green's function of the operator on the left side of the equation. It is more convenient to write a nearby equation with an explicit Green's function, and for this end we find quasi-solutions of the homogeneous equation

$$u'' - (1 + h_0(x))\, u = 0 \tag{35}$$

Formal asymptotic arguments for large x suggest that one solution of (35) has the asymptotic behavior

$$u \sim y_1(x) = e^{-x}\left(1 + \frac{J(x)}{\sqrt{x}}\right), \tag{36}$$

where

$$J(x) = \frac{Se^{-x}}{3} + \frac{S^2e^{-2x}}{16\sqrt{x}} - \frac{19Se^{-x}}{72x} + \frac{S^3e^{-3x}}{108x} - \frac{5S^2e^{-2x}}{48x^{3/2}} + \frac{25S^4e^{-4x}}{20736x^{3/2}} \tag{37}$$

We can readily check that y_1 solves (36) up to $O(x^{-5/2})$ errors:

$$y_1'' - (1 + h_0(x))\, y_1 = q(x)y_1, \tag{38}$$

where

$$q(x)y_1(x) = \sum_{j=5}^{9} x^{-j/2} q_j\left(e^x\right), \tag{39}$$

Here, all $q_j(\zeta)$ are polynomials in $1/\zeta$ of degree at least 2 (see equations (147)-(150) in the Appendix). We chose y_1 to ensure that the error term q_1y_1 is $O(x^{-5/2})$ for large $x \in \Omega_4$.

A second independent solution to the homogeneous equation (38) y_2 is given by $y_1(x)\left\{\int^x \frac{dx'}{y_1^2(x')}\right\}$. With a suitable choice of integration constant, y_2 becomes

$$y_2(x) = y_1(x)\left[\frac{5S^2}{24}\log(ix) + z_2(x)\right],\ \text{where } z_2(x) = z_{2,0}(x) + z_{2,1}(x) + z_{2,R}(x), \tag{40}$$

Here

$$z_{2,0}(x) = \frac{e^{2x}}{2}\ ,\ z_{2,1}(x) = -\frac{2Se^x}{3\sqrt{x}} \tag{41}$$

$$z_{2,R}(x)=\int_{-i\infty}^{x}dx'\left\{\frac{1}{y_1^2(x')}-e^{2x'}+\frac{2Se^{x'}}{3\sqrt{x'}}-\frac{Se^{x'}}{3x'^{3/2}}-\frac{5S^2}{24x'}\right\} \tag{42}$$

Using the fact that y_1 and y_2 defined above solve (38) and their Wronskian is $y_1y_2'-y_2y_1'=1$, inversion of (31) results in the integral equation

$$G(x)=G_0(x)+\int_{-i\infty}^{x}\{y_2(x)y_1(x')-y_1(x)y_2(x')\}\left[-V(x')G(x')+\frac{G^2(x')}{2\sqrt{x'}}\right]dx'=:\mathcal{N}[G](x) \tag{43}$$

where

$$V(x)=V_0(x)+q(x), \tag{44}$$

where

$$V_0(x)=\frac{1}{4x^2} \tag{45}$$

$$G_0(x)=\int_{-i\infty}^{x}\{y_1(x)y_2(x')-y_2(x)y_1(x')\}R(x')dx' \tag{46}$$

Definition 6. Let $\mathcal{S}_4$ be the Banach space of analytic functions in the interior of Ω_4, continuous on $\overline{\Omega_4}$, equipped with the norm

$$\|G\|=\sup_{x\in\Omega_4}\left|x^{5/2}G(x)\right| \tag{47}$$

The usual sup norm will be denoted by $\|.\|_\infty$.

We seek a solution to (43), *i.e.* $G=\mathcal{N}[G]$ in $\mathcal{S}_4$ with $\rho\geqslant 3$. It will be proved that $\mathcal{N}:\mathcal{S}_4\to\mathcal{S}_4$, see (43). The general integral reformulation of (31)

$$G(x)=C_1y_1(x)+C_2y_2(x)+\mathcal{N}[G](x) \tag{48}$$

implies $(C_1,C_2)=(0,0)$ since neither y_1 nor y_2 are in $\mathcal{S}_4$. Therefore, any solution to (31) in $\mathcal{S}_4$ must necessarily satisfy (43).

We now prove the following result:

Theorem 2. *For $\rho\geqslant 3$, there exists unique solution G to the integral equation (43) in a ball $B_4\subset\mathcal{S}_4$ of radius 4. Through the change of variables in* (3) *and* (30)*, this corresponds to the tritronquée solution $y=y_t$.*

The proof of Theorem 2 follows from the following Lemmas that are proved in subsections 4.3-4.5.

Lemma 7. *For $\rho\geqslant 3$ we have*

$$\|G_0\|\leqslant 2 \tag{49}$$

Lemma 8. *For $\rho\geqslant 3$, and $G\in\mathcal{S}_4$,*

$$\left\|\int_{-i\infty}^{x}(y_2(x)y_1(x')-y_1(x)y_2(x'))V(x')G(x')dx'\right\|\leqslant\frac{1}{4}\|G\| \tag{50}$$

Lemma 9. *For $\rho\geqslant 3$, and $G,G_1,G_2\in\mathcal{S}_4$,*

$$\left\|\int_{-i\infty}^{x}(y_2(x)y_1(x')-y_1(x)y_2(x'))\frac{1}{2\sqrt{x'}}G^2(x')dx'\right\|\leqslant\frac{1}{25}\|G\|^2, \tag{51}$$

$$\left\|\int_{-i\infty}^{x}(y_2(x)y_1(x')-y_1(x)y_2(x'))\frac{1}{2\sqrt{x'}}\left[G_1^2(x')-G_2^2(x')\right]dx'\right\| \\ \leqslant\frac{1}{25}\left(\|G_1\|+\|G_2\|\right)\|G_1-G_2\| \tag{52}$$

Proof of Theorem 2 It is clear from Lemmas 7, 8 and 9 that $\mathcal{N} : B_4 \to B_4$ (cf. (43)): for G, G_1 and G_2 in B_4 we have

$$\|\mathcal{N}[G]\| \leqslant \|G_0\| + \frac{1}{4}\|G\| + \frac{1}{25}\|G\|^2 \leqslant 2 + \frac{4}{4} + \frac{16}{25} < 4.$$

and

$$\|\mathcal{N}[G_1] - \mathcal{N}[G_2]\| \leqslant \frac{1}{4}\|G_1 - G_2\| + \frac{8}{25}\|G_1 - G_2\| \leqslant \frac{3}{4}\|G_1 - G_2\|$$

By the contraction mapping theorem, there is a unique solution to (43) in B_4 if $\rho \geqslant 3$. From (30), it is clear that G corresponds to a solution h of (4) that is singularity-free in the closed domain Ω_4 and has the leading order asymptotic behavior $h \sim \frac{Se^{-x}}{\sqrt{x}}$ as $x \to -i\infty$ on the negative imaginary axis. By [1] §5.2 (see also [3]), for any C there is a unique solution h of (4) with the behavior $Ce^{-x}x^{-1/2}$ as $x \to -i\infty$ analytic for large x in a sector in the fourth quadrant (1) The value $C = S$ identifies this solution with the tritronquée. (This also follows from classical results cf. [11].)

4.1. **Preliminary Lemmas.** In this subsection we obtain various integral estimates needed in the sequel.

Definition 10. Let P be a polynomial, $P(\eta) = \sum_{m=m_0}^{m_1} p_m \eta^{-m}$. We define the following weighted ℓ^1 norms:

$$\mathcal{F}_{1,j}[P] = \frac{2}{j-2}\sum_{m=m_0}^{m_l} |p_m| \text{ for } j > 2;\ \mathcal{F}_{2,j}[P] = \sum_{m=m_0}^{m_l} \frac{2}{m}|p_m| \text{ for } m_0 > 0$$

$$\mathcal{F}_{3,j}[P] = \frac{2}{j-3}\sum_{m=m_0}^{m_l} |p_m| \text{ for } j > 3;\ \mathcal{F}_{4,j}[P] = \sum_{m=m_0}^{m_l} \frac{j^2+2j-2}{j(j-1)m}|p_m| \text{ for } j > 1\ ,\ m_0 > 0$$

Lemma 11. *If $l_0 > 2$, g is analytic in Ω_4 with $\|g\|_\infty < \infty$, and*

$$w(x) = \sum_{l=l_0}^{L} x^{-l/2} P_l(e^x), \tag{53}$$

where $P_l(\zeta) = \sum_{m=0}^{m_l} p_{m,l}\zeta^{-m}$ then

$$\Big| \int_{-i\infty}^{x} g(x')w(x')dx' \Big| \leqslant \|g\|_\infty \sum_{l=l_0}^{L} |x|^{-l/2+1} Q_l, \tag{54}$$

where $Q_l = \mathcal{F}_{1,l}[P_l]$

Proof. The various terms in the integrand in (53) are of the form $g(x')p_{m,l}e^{-mx'}x'^{-l/2}$ and thus the contour of integration can be deformed to a radial path from ∞ to $x \in \Omega_4$ and that $|e^{-mx}g(x)| \leqslant \|g\|_\infty$ for $x \in \Omega$. Then, clearly,

$$\Big| p_{m,l}\int_\infty^x g(x')e^{-mx'}x'^{-l/2}dx \Big| \leqslant |p_{m,l}|\|g\|_\infty \int_\infty^{|x|} |x'|^{-l/2} d|x'| \leqslant \frac{|p_{m,l}||x|^{-l/2+1}}{l/2-1}$$

∎

Lemma 12. *If $l_0 > 0$ and*

$$w(x) = \sum_{l=l_0}^{L} x^{-l/2} P_l(e^x), \tag{55}$$

where $P_l(\zeta) = \sum_{m=1}^{m_l} p_{m,l}\zeta^{-m}$ is a polynomial in ζ^{-1}, then

$$\Big| \int_{-i\infty}^{x} w(x')dx' \Big| \leqslant \sum_{l=l_0}^{L} Q_l x^{-l/2} \tag{56}$$

---

(1)The analysis in [1] is done in the first quadrant, but by symmetry w.r.t. $x \in \mathbb{R}$ it applies with straightforward modifications to the fourth quadrant.

where $Q_l = \mathcal{F}_{2,l}\left[P_l\right]$

Proof. We note that the terms in the integral in (56) are of the form

$$p_{m,l} e^{-mx'} {x'}^{-l/2} = \frac{d}{dx'}\left(-\frac{p_{m,l}}{m} e^{-mx'} {x'}^{-l/2}\right) - \frac{l p_{m,l}}{2m} e^{-mx'} {x'}^{-l/2-1}$$

Therefore, integrating out the first term explicitly and deforming the path for the second term to a radial contour, it follows that

$$\left| p_{m,l} \int_{\infty}^{x} e^{-mx'} {x'}^{-l/2} dx' \right| \leqslant \frac{2|p_{m,l}|}{m} |x|^{-l/2}$$

∎

Lemma 13. *If for* $l_0 > 3$ $w(x) = \sum_{l=l_0}^{L} x^{-l/2} P_l(e^x)$, *where* $P_l(\zeta) = \sum_{m=0}^{m_l} p_{m,l} \zeta^{-m}$, *and if* g *is analytic in* Ω_4 *with* $\|g\|_\infty < \infty$, *then*

$$\Big| \int_{-i\infty}^{x} \log \frac{x'}{x}\, g(x') w(x') dx' \Big| \leqslant \|g\|_\infty \sum_{l=l_0}^{L} Q_l |x|^{-l/2+1},$$

where $Q_l = \mathcal{F}_{3,l}\left[P_l\right]$

Proof. Once again because of the analyticity and decay of integrand, we may deform the integration path to a radial one joining ∞ to $x \in \Omega_4$. The general term in the integrand is of the form

$$p_{m,l} {x'}^{-l/2} e^{-mx'} g(x') \log \frac{x'}{x} dx'$$

Since it is readily checked that for $|x'| \geqslant |x|$,

$$\Big| \log \frac{x'}{x} \Big| \leqslant \left(\frac{|x'|}{|x|} \right)^{1/2},$$

it follows that

$$\left| \int_{\infty}^{x} p_{m,l} {x'}^{-l/2} e^{-mx'} g(x') \log \frac{x'}{x} dx' \right| \tag{57}$$
$$\leqslant \|g\|_\infty |p_{m,l}| |x|^{-1/2} \int_{\infty}^{|x|} |x'|^{-l/2+1/2} d|x'| = \|g\|_\infty \frac{2|p_{m,l}|}{l-3} |x|^{-l/2+1}$$

∎

Lemma 14. *If for* $l_0 > 1$, $w(x) = \sum_{l=l_0}^{L} x^{-l/2} P_l(e^x)$, *where* $P_l(\zeta) = \sum_{m=1}^{m_l} p_{m,l} \zeta^{-m}$, *then*

$$\left| \int_{-i\infty}^{x} \log \frac{x'}{x} w(x') dx' \right| \leqslant \sum_{l=l_0}^{L} Q_l |x|^{-l/2},$$

where $Q_l = \mathcal{F}_{4,l}\left[P_l\right]$

Proof. Once again because of the analyticity and decay of the integrand, we may deform the integration path to a radial one joining ∞ to $x \in \Omega_4$. A general term in the integrand is of the form

$$p_{m,l} {x'}^{-l/2} e^{-mx'} \log \frac{x'}{x} = \frac{d}{dx'}\left(-\frac{p_{m,l}}{m} e^{-mx'} {x'}^{-l/2} \log \frac{x'}{x} \right) + \frac{p_{m,l}}{m} e^{-mx'} {x'}^{-l/2-1}$$
$$-\frac{l}{2m} p_{m,l} e^{-mx'} {x'}^{-l/2-1} \log \frac{x'}{x}$$

Noting that the complete derivative is zero at the end point $x' = x$ and applying Lemma 13 to bound the integral of the third term (on a radial path) we immediately obtain

$$\Big| \int_{\infty}^{x} p_{m,l} e^{-mx'} \log \frac{x'}{x} dx' \Big| \leqslant \frac{1}{m} |p_{m,l}| \left(\frac{2}{l} + \frac{l}{l-1} \right) |x|^{-l/2},$$

from which the Lemma follows. ∎

4.2. **Bounds on J, y_1, $z_{2,R}$ and z_2 for $x \in \Omega_4$.** Let

$$y_{1,0}(x) = e^{-x}\ ,\ y_{1,1}(x) = \frac{S}{3\sqrt{x}}e^{-2x}\ ,\ y_{1,R} = y_1 - y_{1,0} - y_{1,1}, \tag{58}$$

and define

$$j(x) = \frac{3S}{16}\mathrm{e}^{-x} + \left(-\frac{19}{24} + \frac{S^2 e^{-2x}}{36}\right)x^{-1/2} + \left(-\frac{5}{16}\,S\mathrm{e}^{-x} + \frac{25}{6912}\,S^3\mathrm{e}^{-3\,x}\right)x^{-1} \tag{59}$$

comparing with (37) we see that

$$J(x) = \frac{Se^{-x}}{3}\left(1 + \frac{j(x)}{\sqrt{x}}\right) \tag{60}$$

Note that for $x \in \Omega_4$,

$$|j(x)| \leqslant \frac{3|S|}{16} + \frac{19}{24}\frac{1}{\sqrt{\rho}} + \frac{|S|^2}{36\sqrt{\rho}} + \frac{5}{16}\frac{|S|}{\rho} + \frac{25|S|^3}{6912\rho} =: j_m \tag{61}$$

Using $J(x)$ in (37), it follows that for $x \in \Omega_4$ we have

$$\left|e^x J(x)\right| \leqslant \frac{|S|}{3}\left(1 + \frac{j_m}{\sqrt{\rho}}\right) =: J_M \tag{62}$$

From (36), it follows that

$$\left|e^x y_1\right| \leqslant \left(1 + \frac{J_M}{\sqrt{\rho}}\right) =: Y_{1,M} \tag{63}$$

Now, (37) and (58) imply

$$\left|xe^{2x}y_{1,R}(x)\right| \leqslant \frac{|S|}{3}j_m =: Y_{1,R,M} \tag{64}$$

From (58), it also follows that for $x \in \Omega_4$,

$$\left|e^x\left(y_{1,0}(x) + y_{1,1}(x)\right)\right| \leqslant 1 + \frac{|S|}{3\sqrt{\rho}} \tag{65}$$

Expressing y_1 in terms of J, as in (36) and (37), in (42), it follows that

$$\begin{aligned} z_{2,R}(x) = z_{2,R,0}(x) + \int_{-i\infty}^{x} E(x')dx' + \frac{7S}{36}\int_{-i\infty}^{x}\frac{e^{x'}}{x'^{3/2}}dx' \\ + \int_{-i\infty}^{x} e^{2x'}\left\{\left(1 + \frac{J(x')}{\sqrt{x'}}\right)^{-2} - 1 + \frac{2J(x')}{\sqrt{x'}} - \frac{3J^2(x')}{x}\right\}dx', \end{aligned} \tag{66}$$

where

$$z_{2,R,0}(x) = \frac{23S^2}{72x} - \frac{361S^2}{3456x^2} - \frac{23S^3e^{-x}}{216x^{3/2}} - \frac{577S^4e^{-2x}}{41472x^2}\ , \tag{67}$$

$$E(x) = \sum_{j=5}^{8} x^{-j/2}E_j\left(e^x\right), \tag{68}$$

where each $E_j(\zeta)$ is a polynomial in $1/\zeta$ with no constant term (the precise expressions are given in (151)-(152) in the appendix). We note that

$$\left|xz_{2,R,0}\right| \leqslant \frac{23|S|^2}{72} + \frac{23|S|^3}{216\rho^{1/2}} + \left(\frac{361|S|^2}{3456} + \frac{577|S|^4}{41472}\right)\rho^{-1} \tag{69}$$

Using Lemma 12, it follows that

$$\left|x\int_{-i\infty}^{x} E(x')dx'\right| \leqslant \sum_{j=5}^{8}\rho^{-j/2+1}\mathcal{F}_{2,j}\left[E_j\right] =: E_M \tag{70}$$

(see Definition 10 and the expression of E_M in the Appendix, (153)). On integration by parts, we get

$$\frac{7Se^{-x}}{36}\int_{-i\infty}^{x}\frac{e^{x'}}{x'^{3/2}}dx' = \frac{7S}{36x^{3/2}} + \frac{7S}{24}\int_{-i\infty}^{x}\frac{e^{x'-x}}{x'^{5/2}}dx' \tag{71}$$

Therefore for $x \in \Omega_4$ we get

$$\left|\frac{7Sxe^{-x}}{36}\int_{-i\infty}^{x}\frac{e^{x'}}{x'^{3/2}}dx'\right| \leqslant \frac{7|S|\left(\sqrt{2}+1\right)}{36\sqrt{\rho}} \tag{72}$$

where we used the fact that on a vertical contour connecting $-i\infty$ to $x \in \Omega_4$, $|dx'| \leqslant \sqrt{2}d|x'|$. We note that

$$e^{2x}\left\{\left(1+x^{-1/2}J\right)^{-2} - 1 + 2x^{-1/2}J - 3x^{-1}J^2\right\} = -4e^{2x}x^{-3/2}J^3 + \frac{5x^{-2}e^{2x}J^4}{1+x^{-1/2}J} - \frac{x^{-5/2}e^{2x}J^5}{(1+x^{-1/2}J)^2}, \tag{73}$$

and

$$-4e^{2x}x^{-3/2}J^3(x) = -\frac{4S^3e^{-x}}{27x^{3/2}} - \frac{4S^3}{27x^{3/2}}e^{-x}\left[\left(1+\frac{j(x)}{\sqrt{x}}\right)^3 - 1\right] \tag{74}$$

Now, using Lemma 12, it follows that

$$\left|x\int_{-i\infty}^{x}\frac{4S^3e^{-x'}}{27x'^{3/2}}dx'\right| \leqslant \frac{8|S|^3}{27\rho^{1/2}}, \tag{75}$$

Deforming the contour to a radial one, it is clear that

$$\left| -\frac{4S^3x}{27}\int_{-i\infty}^{x}\frac{e^{-x'}}{x'^{3/2}}\left[\left(1+\frac{j(x')}{\sqrt{x'}}\right)^3 - 1\right]dx'\right| \leqslant \frac{4|S|^3}{27}\left(3j_m + 3\frac{j_m^2}{\sqrt{\rho}} + \frac{j_m^3}{\rho}\right) \tag{76}$$

Therefore, using (75) and (76) in (74) we get

$$\left|x\int_{-i\infty}^{x}\frac{-4e^{2x'}J^3(x')}{x'^{3/2}}dx'\right| \leqslant \frac{8|S|^3}{27\sqrt{\rho}} + \frac{4|S|^3j_m}{9}\left(1+\frac{j_m}{\sqrt{\rho}}+\frac{j_m^2}{3\rho}\right) \tag{77}$$

From (73) and (77), it follows that

$$\begin{aligned}(78)\quad &\left|x\int_{-i\infty}^{x}\left\{e^{2x'}\left(1+x'^{-1/2}J(x')\right)^{-2} - 1 + 2x'^{-1/2}J(x') - 3x'^{-1}J^2(x')dx'\right\}\right| \\ &\leqslant \frac{8|S|^3}{27\sqrt{\rho}} + \frac{4|S|^3j_m}{9}\left(1+\frac{j_m}{\sqrt{\rho}}+\frac{j_m^2}{3\rho}\right) + \frac{5J_M^4}{1-\rho^{-1/2}J_M} + \frac{2\rho^{-1/2}J_M^5}{3\left(1-\rho^{-1/2}J_M\right)^2}\end{aligned}$$

Therefore, combining all the estimates (69), (70), (72) and (78) in the expression of $z_{2,R}$ in (66) we get

$$\begin{aligned}(79)\quad &\left|xe^{-x}z_{2,R}\right| \leqslant \frac{23|S|^2}{72} + \frac{23|S|^3}{216\rho^{1/2}} + \left(\frac{361|S|^2}{3456} + \frac{577|S|^4}{41472}\right)\rho^{-1} + E_M \\ &+\frac{7|S|\left(\sqrt{2}+1\right)}{36\sqrt{\rho}} + \frac{8|S|^3}{27\sqrt{\rho}} + \frac{4|S|^3j_m}{9}\left(1+\frac{j_m}{\sqrt{\rho}}+\frac{j_m^2}{3\rho}\right) + \frac{5J_M^4}{1-\rho^{-1/2}J_M} + \frac{2\rho^{-1/2}J_M^5}{3\left(1-\rho^{-1/2}J_M\right)^2} =: z_{2,R,M}\end{aligned}$$

Therefore, using (40) and (41) we get

$$|e^{-2x}z_2| \leqslant \frac{1}{2} + |e^{-x}z_{2,1}| + \frac{z_{2,R,M}}{\rho} = \frac{1}{2} + \frac{2|S|}{3\sqrt{\rho}} + \frac{z_{2,R,M}}{\rho} =: z_{2,M} \tag{80}$$

To help the reader who would like to check the intermediate steps in the calculations, we provide in the Appendix the numerical values for $\rho = 3$ of the various constants appearing in the estimates.

4.3. **Bounds on $V(x)$ and proof of Lemma 8.** We first seek bounds on qy_1. It is clear from the expression of qy_1 in (39) that Lemma 11 applies to $w(x) = x^{-5/2}e^{2x}qy_1$, $g(x) = x^{5/2}G(x)$; noting that $\|G\| = \|x^{5/2}G\|_\infty$, we obtain

$$\left|x^{5/2}\int_{-i\infty}^{x} e^{2x'}q(x')y_1(x')G(x')dx'\right| \leqslant \|G\| \sum_{j=10}^{14} \mathcal{F}_{1,j}\left[q_{j-5}\right]\rho^{-j/2+7/2} = M_q\|G\| \tag{81}$$

(see Def. 10). The explicit formula of M_q is given in (154) in the Appendix. Further,

$$\left|x^{5/2}y_1(x)\int_{-i\infty}^{x}\left(z_2(x') - z_2(x)\right)q(x')y_1(x')G(x')dx'\right| \leqslant 2Y_{1,M}z_{2,M}M_q\|G\| \tag{82}$$

In (82), recalling that $q(x)y_1(x)$ is a polynomial in $1/e^x$ of degree at least 2, we applied Lemma 11 with $g(x') = e^{-2x'}z_2(x')x'^{5/2}G(x')$ and $w(x') = x'^{-5/2}e^{2x'}q(x')y_1(x')$ in the part of the integral involving $z_2(x')$, while in the second one, we took $g(x') = e^{x-x'}x'^{5/2}G(x')$, $w(x') = x'^{-5/2}e^{x'}q_1(x')y_1(x')$ and used $\left|e^{-x}z_2(x)y_1(x)\right| \leqslant z_{2,M}Y_{1,M}$. Lemma 13, for $w(x) = x^{-5/2}q(x)y_1(x)$, $g(x) = x^{5/2}G$, implies

$$\begin{aligned}\left|\frac{5S^2}{24}x^{5/2}y_1(x)\int_{-i\infty}^{x}\log\frac{x'}{x}\,q(x')y_1(x')G(x')dx'\right| &\leqslant \frac{5|S|^2}{24}\|G\|\sum_{j=10}^{14}\mathcal{F}_{3,j}\left[q_{j-5}\right]\rho^{-j/2+7/2} \\ &=: \frac{5|S|^2}{24}M_{L,q}\|G\|\end{aligned} \tag{83}$$

where the detailed expression of $M_{L,q}$ is given in the Appendix, (157)). Therefore, using (40), (82) and (83), it follows that

$$\left\|\int_{-i\infty}^{x}\left(y_2(x)y_1(x') - y_1(x)y_2(x')\right)q(x')G\right\| \leqslant Y_{1,M}\left(2z_{2,M}M_q + \frac{5|S|^2}{24}M_{L,q}\right)\|G\| \tag{84}$$

We now bound the terms involving $V_0 = \frac{1}{4x^2}$. Noting Lemma 11 applies to $w(x) = e^{-x}V_0x^{-5/2}$ and $g(x) = x^{5/2}e^xGy_1$; since both $\|G\| = \|x^{5/2}G\|_\infty$ and $|e^xy_1| \leqslant Y_{1,M}$, it follows that

$$\left|x^{5/2}\int_{-i\infty}^{x}V_0(x')y_1(x')G(x')dx'\right| \leqslant \frac{Y_{1,M}}{14\rho}\|G\| \tag{85}$$

Since e^xy_1, $e^{-2x}z_2$ and $x^{5/2}G$ are bounded by $Y_{1,M}$, $z_{2,M}$ and $\|G\|$ respectively, we have

$$\begin{aligned}&\left|x^{5/2}e^{-x}\int_{-i\infty}^{x}V_0(x')y_1(x')z_2(x')G(x')\right| \\ &\qquad \leqslant Y_{1,M}z_{2,M}\|G\|\left\{|x|^{5/2}\int_{-i\infty}^{x}|e^{-x+x'}|\frac{1}{4|x'|^{9/2}}|dx'|\right\} \leqslant \frac{\sqrt{2}}{14\rho}Y_{1,M}z_{2,M}\|G\|,\end{aligned} \tag{86}$$

where we used the fact that on a vertical contour joining $-i\infty$ to $x \in \Omega_4$ we have $|e^{x'-x}| = 1$ and $|dx'| \leqslant \sqrt{2}d|x'|$. Lemma 13 applied to $w(x) = e^{-x}x^{-5/2}V_0$, $g(x) = e^xy_1x^{5/2}G$ gives

$$\left|\frac{5S^2}{24}x^{5/2}\int_{-i\infty}^{x}\log\frac{x'}{x}V_0(x')y_1(x')G(x')\right| \leqslant \frac{5|S|^2}{288\rho}\|G\|Y_{1,M} \tag{87}$$

Therefore, combining (85)-(87) and using (40), we obtain

$$\left|x^{3/2}\int_{-i\infty}^{x}\left(y_2(x)y_1(x') - y_1(x)y_2(x')\right)V_0(x')G(x')\right| \leqslant Y_{1,M}^2\left(\frac{\sqrt{2}+1}{14\rho}z_{2,M} + \frac{5|S|^2}{288\rho}\right)\|G\| \tag{88}$$

Collecting the contributions of the terms involving q and V_0 in (84) and (88) respectively, it follows that

$$\begin{aligned}(89)\quad &\left\|\int_{-i\infty}^{x}\left(y_2(x)y_1(x')-y_1(x)y_2(x')\right)V(x')G(x')dx'\right\| \\ &\qquad\leqslant Y_{1,M}\left\{\left(2z_{2,M}M_q+\frac{5|S|^2}{24}M_{L,q}\right)+Y_{1,M}\left(\frac{\sqrt{2}+1}{14\rho}z_{2,M}+\frac{5|S|^2}{288\rho}\right)\right\}\|G\| \\ &\qquad\qquad =: V_M\|G\|\end{aligned}$$

Since all the quantities involved in V_M are decreasing in ρ, it is clear that for $\rho \geqslant 3$, V_M is bounded by its value at $\rho = 3$, which in turn is less than $9/40$ and Lemma 8 follows.

4.4. **Nonlinear terms and proof of Lemma 9.** Applying Lemma 11 to $w(x') = e^{-x'}x'^{-11/2}$, $g(x') = \frac{1}{2}e^{x-x'}x'^5G^2(x')e^{x'}y_1(x')$, noting that $|e^{x-x'}| \leqslant 1$ on a radial contour in Ω_4, and finally that $|x'^5G^2(x')| \leqslant \|G\|^2$, $|e^{x'}y_1(x')| \leqslant Y_{1,M}$, we obtain

$$(90)\qquad \left|e^x y_1(x)e^{-2x}z_2(x)x^{5/2}\int_{-i\infty}^{x}e^{x-x'}\frac{e^{x'}y_1(x')G^2(x')}{2\sqrt{x'}}dx'\right| \leqslant \frac{Y_{1,M}^2 z_{2,M}}{9\rho^2}\|G\|^2$$

Furthermore, we note that

$$(91)\qquad \left|e^x y_1(x)x^{5/2}\int_{-i\infty}^{x}e^{-x+x'}\frac{e^{x'}y_1(x')e^{-2x'}z_2(x')G^2(x')}{2\sqrt{x'}}dx'\right| \leqslant \frac{\sqrt{2}Y_{1,M}^2 z_{2,M}}{9\rho^2}\|G\|^2,$$

where in (91) we have $|dx'| \leqslant \sqrt{2}d|x'|$ and $|e^{x'-x}| = 1$ on the vertical contour joining $-i\infty$ to $x \in \Omega_4$. Applying Lemma 13 to $w(x) = e^{-x}x^{-11/2}$, $g(x) = x^5G^2e^xy_1(x)$, we get

$$(92)\qquad \left|\frac{5S^2}{24}\, y_1(x)x^{5/2}\int_{-i\infty}^{x}\log\frac{x'}{x}\frac{y_1(x')G^2(x')}{2\sqrt{x'}}dx'\right| \leqslant \frac{5|S|^2Y_{1,M}^2}{192\rho^2}\|G\|^2$$

Combining (90), (91) and (92) we obtain

$$\begin{aligned}(93)\quad &\left\|\int_{-i\infty}^{x}\left(y_2(x)y_1(x')-y_1(x)y_2(x')\right)\frac{G^2(x')}{2\sqrt{x'}}dx'\right\| \leqslant \frac{Y_{1,M}^2}{\rho^2}\left(\left[\frac{1+\sqrt{2}}{9}\right]z_{2,M}+\frac{5|S|^2}{192}\right)\|G\|^2 \\ &\qquad\qquad =: T_M\|G\|^2\end{aligned}$$

A very similar calculation shows that

$$\left\|\int_{-i\infty}^{x}\left(y_2(x)y_1(x')-y_1(x)y_2(x')\right)\frac{G_1^2(x')-G_2^2(x')}{2\sqrt{x'}}dx'\right\| \leqslant T_M\left(\|G_1\|+\|G_2\|\right)\|G_1-G_2\|$$

Proof of Lemma 9 follows since a calculation of T_M, which is a decreasing function of ρ, shows $T_M \leqslant \frac{18}{467} < \frac{1}{25}$ for $\rho \geqslant 3$.

4.5. **Bounds involving G_0 and proof of Lemma 7.** Using (32)-(33) and the form of y_2 in (40), it is convenient to decompose G_0 defined in (46) as follows

$$(94)\qquad G_0 = G_{0,1}+G_{0,2}+G_{0,3}+G_{0,4}+G_{0,5}+G_{0,6}+G_{0,7}$$

where

$$(95)\qquad G_{0,1}(x) = y_1(x)\int_{-i\infty}^{x}y_1(x')\left[z_2(x')-z_2(x)\right]\ \tilde{R}(x')dx'$$

$$(96)\qquad G_{0,2}(x) = y_1(x)\int_{-i\infty}^{x}y_{1,R}(x')\left[z_2(x')-z_2(x)\right]\left(R_0(x')+R_1(x')\right)dx'$$

$$(97)\qquad G_{0,3}(x) = y_1(x)\int_{-i\infty}^{x}\left[y_{1,0}(x')+y_{1,1}(x')\right]\left[z_{2,R}(x')-z_{2,R}(x)\right]\left(R_0(x')+R_1(x')\right)dx'$$

$$G_{0,4}(x) = y_1(x)\int_{-i\infty}^{x} \left[y_{1,0}(x') + y_{1,1}(x')\right]\{z_{2,0}(x') + z_{2,1}(x') \\ -z_{2,0}(x) - z_{2,1}(x)\}\left(R_0(x') + R_1(x')\right)dx' \tag{98}$$

$$G_{0,5}(x) = \frac{5S^2}{24}y_1(x)\int_{-i\infty}^{x} \log\frac{x'}{x}\; y_1(x')\tilde{R}(x')dx' \tag{99}$$

$$G_{0,6}(x) = \frac{5S^2}{24}y_1(x)\int_{-i\infty}^{x} \log\frac{x'}{x}\; y_{1,R}(x')\left[R_0(x') + R_1(x')\right]\; dx' \tag{100}$$

$$G_{0,7}(x) = \frac{5S^2}{24}y_1(x)\int_{-i\infty}^{x} \log\frac{x'}{x}\left(y_{1,0}(x') + y_{1,1}(x')\right)\left(R_0(x') + R_1(x')\right)dx' \tag{101}$$

In §4.5.1 below we obtain bounds M_j for $\|G_{0,j}\|$ for $j = 1, 2..7$; using those, we get

$$\|G_0\| \leqslant \sum_{j=1}^{7} M_j \tag{102}$$

The formulas for $M_j, j = 1, ..., 7$ are given in the following subsections. These expressions will be shown to be decreasing in ρ, and Lemma 7 will follow using the values of M_j at $\rho = 3$.

4.5.1. *Bounds on* $G_{0,1}$. Using Lemma 11, with $g(x') = e^{x-x'}e^{x'}y_1(x')$ and $w(x') = e^{-x'}\tilde{R}(x')$ we get

$$\left|x^{5/2}y_1(x)z_2(x)\int_{-i\infty}^{x} y_1(x')\tilde{R}(x')dx'\right| \leqslant Y_{1,M}^2 z_{2,M}\sum_{j=7}^{9}\mathcal{F}_{1,j}\left[r_j\right]\rho^{-j/2+7/2} \tag{103}$$

Note in the estimate above, the factor e^x outside the integral was placed inside the integral in $g(x')$; this is legitimate since in a radial deformation of the integration path from ∞ to $x \in \Omega_4$ we have $|e^{x-x'}| \leqslant 1$.

Noting again that $\tilde{R}(x) + \frac{392}{625x^{7/2}}$ has only decaying exponentials, Lemma 11 (this time with $g(x) = e^{-x}y_1(x)z_2(x)$ and $w(x) = e^x\left(\tilde{R}(x) + \frac{392}{625}x^{-7/2}\right)$ implies

$$\left|x^{5/2}y_1(x)\int_{-i\infty}^{x} y_1(x')z_2(x')\left[\tilde{R}(x') + \frac{392}{625x'^{7/2}}\right]dx'\right| \leqslant Y_{1,M}^2 z_{2,M}\sum_{j=7}^{9}\mathcal{F}_{1,j}\left[\tilde{r}_j\right]\rho^{-j/2+7/2} \tag{104}$$

Since on a vertical contour in joining $-i\infty$ to $x \in \Omega_4$, $|e^{-x+x'}| = 1$ and $|dx'| \leqslant \sqrt{2}d|x'|$, we get

$$\left|x^{5/2}y_1(x)\int_{-i\infty}^{x} y_1(x')z_2(x')\frac{392}{625x'^{7/2}}dx'\right| \leqslant \frac{784}{3125}Y_{1,M}^2 z_{2,M}\sqrt{2} \tag{105}$$

It follows that

$$\|G_{0,1}\| \leqslant Y_{1,M}^2 z_{2,M}\left\{\frac{784}{3125}\sqrt{2} + \mathcal{F}_{1,7}\left[r_7 + \tilde{r}_7\right] + 2\sum_{j=8}^{9}\mathcal{F}_{1,j}\left[r_j\right]\right\} =: Y_{1,M}^2 z_{2,M}M_{G,1} =: M_1 \tag{106}$$

where the explicit expression of $M_{G,1}$ is given in (164) in the Appendix.

4.5.2. *Bounds on* $G_{0,2}$. From (64), (80) we note that $e^{2x}xy_{1,R}$ and $e^{-2x}z_2(x)$ are bounded by $Y_{1,R,M}$ and $z_{2,M}$ respectively. Lemma 11 applied to $w(x) = \frac{1}{x}\left(R_0 + R_1\right)$ implies

$$\|G_{0,2}\| \leqslant 2Y_{1,M}z_{2,M}Y_{1,R,M}\sum_{j=7}^{8}\rho^{-j/2+7/2}\mathcal{F}_{1,j}\left[r_{j-2}\right] =: 2Y_{1,M}z_{2,M}Y_{1,R,M}M_{G,2} := M_2 \tag{107}$$

where the formula for $M_{G,2}$ is given in (168) in the Appendix.

4.5.3. *Bounds on $G_{0,3}$:* Note that $xe^{-x}z_{2,R}$ and $e^x\left(y_{1,0}+y_{1,1}\right)$ are bounded by $z_{2,R,M}$ and $1+\frac{|S|}{3\sqrt{\rho}}$ resp. Lemma 11 applied to $w(x')=\left[R_0(x')+R_1(x')\right]/x'$ for the term containing $z_{2,R}(x')$ and to $w(x')=\left[R_0(x')+R_1(x')\right]$ for the one containing $z_{2,R}(x)$ implies

$$\|G_{0,3}\| \leqslant Y_{1,M} z_{2,R,M}\left(1+\frac{|S|}{3\sqrt{\rho}}\right) M_{G,3} =: M_3 \tag{108}$$

where

$$M_{G,3} = M_{G,2} + \sum_{j=5}^{6} \rho^{-j/2+5/2} \mathcal{F}_{1,j}\left[r_j\right]. \tag{109}$$

The concrete expression of $M_{G,3}$ is given in (169) in the Appendix.

4.5.4. *Bounds on $G_{0,4}$:* Using (33) and (58), it follows that

$$T(x) =: \left(y_{1,0}+y_{1,1}\right)\left(R_0+R_1\right) = \sum_{j=5}^{7} x^{-j/2} t_j(e^x), \tag{110}$$

where $t_j(\zeta)$ are polynomials in $1/\zeta$, having no constant and linear terms; the precise expressions are in (170)-(171) in the Appendix.

$$U(x) =: \left(y_{1,0}+y_{1,1}\right)\left(z_{2,0}+z_{2,1}\right)\left(R_0+R_1\right) = \sum_{j=5}^{8} x^{-j/2} u_j(e^x) \tag{111}$$

where $u_j(\zeta)$ are polynomials in $1/\zeta$ without constant terms–see (172)-(173) in the Appendix. We also note that

$$T(x) = \frac{d}{dx}\left[\sum_{j=5}^{7} x^{-j/2}\ \tau_j\left(e^x\right)\right] + \sum_{j=7}^{9} x^{-j/2}\ \tilde{t}_j(e^x), \tag{112}$$

where $\tau_j(\zeta)$, $\tilde{t}_j(\zeta)$, are polynomials in $1/\zeta$, see (174)-(177) in the Appendix, with no constant or linear terms. Again, we note that

$$U(x) = \frac{d}{dx}\left[\sum_{j=5}^{8} x^{-j/2}\nu_j\left(e^x\right)\right] + \sum_{j=7}^{10} x^{-j/2}\tilde{u}_j\left(e^x\right) \tag{113}$$

where the polynomials in $1/\zeta$, $\tilde{u}_j(\zeta)$, $\nu_j(\zeta)$ have no constant terms, see (178)-(181). Using (98), (110)-(113), it follows that

$$\begin{aligned} G_{0,4}(x) = {} & y_1(x)\left\{\sum_{j=5}^{8} x^{-j/2}\nu_j\left(e^x\right) - \left(z_{2,0}(x)+z_{2,1}(x)\right)\sum_{j=5}^{7} x^{-j/2}\tau_j\left(e^x\right)\right\} \\ & + y_1(x)\int_{-i\infty}^{x}\left\{\sum_{j=7}^{10} {x'}^{-j/2}\tilde{u}_j\left(e^{x'}\right) - \left[z_{2,0}(x)+z_{2,1}(x)\right]\sum_{j=7}^{9} {x'}^{-j/2}\tilde{t}_j\left(e^{x'}\right)\right\} dx' \end{aligned} \tag{114}$$

Straightforward calculations show that

$$\sum_{j=5}^{8} x^{-j/2}\nu_j\left(e^x\right) - \left(z_{2,0}(x)+z_{2,1}(x)\right)\sum_{j=5}^{7} x^{-j/2}\tau_j\left(e^x\right) = \sum_{j=5}^{8} x^{-j/2} p_j\left(e^x\right), \tag{115}$$

where $p_j(\zeta)$s are also polynomials in $1/\zeta$ with no constant term; see the Appendix, starting with eq. (182).

Applying Lemma 11 to the two terms in second integral on the right of (114), using $|e^{x-x'}| \leqslant 1$, $w(x)=\sum_{j=7}^{9} x^{-j/2}\tilde{t}_j\left(e^x\right)$ for the integral on the first line and $w(x)=\sum_{j=7}^{10} x^{-j/2}\tilde{u}_j\left(e^x\right)$ in the

second line we get

$$\|G_{0,4}\| \leqslant Y_{1,M}\left[\sum_{j=5}^{8}\rho^{-j/2+5/2}\tilde{p}_j(1)\right] + Y_{1,M}\left\{\left(\frac{1}{2}+\frac{2|S|}{3\sqrt{\rho}}\right)\sum_{j=7}^{9}\rho^{-j/7+7/2}\mathcal{F}_{1,j}\left[\tilde{t}_j\right] + \sum_{j=7}^{10}\rho^{-j/7+7/2}\mathcal{F}_{1,j}\left[\tilde{u}_j\right]\right\} =: Y_{1,M}\left(M_{G,4,0}+M_{G,4,1}\right) =: M_4, \tag{116}$$

see (184), (185), where $\tilde{p}_j$ is the polynomial obtained from p_j by replacing each coefficient by its absolute value.

4.5.5. *Bounds on $G_{0,5}$, $G_{0,6}$ and $G_{0,7}$.* For $G_{0,5}$ we simply use Lemma 13 with $w(x) = e^{-x}\tilde{R}(x)$, and $g(x) = e^x y_1$. We obtain

$$\left|x^{5/2}\frac{5S^2}{24}y_1\int_{-i\infty}^{x}\log\frac{x'}{x}y_1(x')\tilde{R}(x')\right| \leqslant \frac{5|S|^2}{24}Y_{1,M}^2\sum_{j=7}^{9}|x|^{-j/2+7/2}\mathcal{F}_{3,j}\left[r_j\right] =: \frac{5|S|^2}{24}Y_{1,M}^2 M_{G,5} =: M_5 \tag{117}$$

where $M_{G,5}$ is given in (186) in the Appendix. We again use Lemma 13 with $w(x) = \frac{e^{-x}}{x}\left(R_0+R_1\right)$ and $g(x) = e^x x y_{1,R}$, to obtain

$$\left|x^{5/2}\frac{5S^2}{24}y_1\int_{-i\infty}^{x}\log\frac{x'}{x}y_{1,R}(x')\left(R_0(x')+R_1(x')\right)\right| \leqslant \frac{5|S|^2}{24}Y_{1,M}Y_{1,R,M}\sum_{j=7}^{9}\rho^{-j/2+7/2}\mathcal{F}_{3,j}\left[r_{j-2}\right] =: \frac{5|S|^2}{24}Y_{1,M}Y_{1,R,M}M_{G,6} =: M_6, \tag{118}$$

where $M_{G,6}$ is given in (187) in the Appendix. Now consider $G_{0,7}(x)$. Recall that

$$T(x) = \left(y_{1,0}+y_{1,1}\right)\left(R_0+R_1\right) = \sum_{j=5}^{7}x^{-j/2}t_j\left(e^x\right) \tag{119}$$

From the expressions of t_j in (170)- (171), it is clear that (119) involves only decaying exponentials. Therefore, we may apply Lemma 14 to $w(x) = T(x)$ giving

$$\left|x^{5/2}\frac{5S^2}{24}y_1(x)\int_{-i\infty}^{x}\log\frac{x'}{x}\left(y_{1,0}(x')+y_{1,1}(x')\right)\left(R_0(x')+R_1(x')\right)dx'\right| \leqslant \frac{5|S|^2}{24}Y_{1,M}\sum_{j=5}^{7}\rho^{-j/2+5/2}\mathcal{F}_{4,j}\left[t_j\right] =: \frac{5|S|^2}{24}Y_{1,M}M_{G,7} =: M_7 \tag{120}$$

where $M_{G,7}$ is given in (188) in the Appendix.

5. Analysis of y_t for $|z| \leqslant 1.7$

For the analysis of the inner region $|z| < 17/10$ it is convenient to use the symmetry of the equation and write it as

$$g'' = 6g^2 + t \text{ where } g(t) = e^{2\pi i/5}y(-te^{\pi i/5}) \tag{121}$$

We take initial conditions close to (19), (20), converted into conditions for $g(t)$

$$g(t_0) = -\frac{280}{519};\quad g'(t_0) = \frac{150}{1013};\text{ where } t_0 = -1.7 \tag{122}$$

We first construct a polynomial which is sufficiently close to g in $L^\infty([t_0,0])$ so as to be able to rigorously preserve error bounds of the order of those in (18). The way the polynomial g_0 below was obtained is essentially by projecting g, calculated from its truncated Taylor series, on Chebyshev polynomials of order seven, enough for the aforementioned goal; the polynomial is

$$g_0(t) = -\frac{280}{519}+\frac{150s}{1013}+\frac{239s^2}{10331}+\frac{110s^3}{14779}-\frac{32s^4}{9853}+\frac{9s^5}{4397}-\frac{16s^6}{39505}+\frac{8s^7}{49105},\text{ where } s = t-t_0. \tag{123}$$

Definition 15. For a polynomial $P(x) = \sum_{k=0}^{n} c_k x^k$ on an interval I we define an ℓ^1 norm by $\|P\|_1 = \sum_{k=0}^{n} |c_k| m^k$ where $m = \sup_I |x|$.

Taking $g = g_0 + \delta$ we get

$$\delta'' - 12 g_0 \delta = 6\delta^2 + R;\ \delta(0) = a_1 = g(0) - g_0(0), \delta'(0) = a_2 = g'(0) - g_0'(0) \tag{124}$$

where $-R = g_0'' - 6g_0^2 - t$ is a polynomial of degree 14.

Proposition 16. *For $t \in [t_0, 0]$ we have $|R| < 1/8619$.*

Note 4. *(a)* Estimating rigorously and with good accuracy, a polynomial $P(x)$ on an interval $I = [a, b]$ is elementary, and it can be done rigorously and efficiently in a number of ways. The one used here is the same as in [4]. We choose a suitable partition of $[a, b]$, $\mathbf{\Pi} = \langle x_0, x_1, ..., x_{n-1}, x_n\rangle$, where $x_0 = a$, $x_n = b$; then write $t = \frac{1}{2}(x_i + x_{i-1}) + u$ on each subinterval $[x_{i-1}, x_i]$ for $i = 1, ..n$ and re-expand P to obtain a polynomial in u. The polynomial in u is estimated by taking the extremum of the modulus of the cubic subpolynomial and placing $\|\cdot\|_1$ on the rest. In practice a small number of partition points suffices to get a good bound.

Since all partition points are nonpositive, to simplify the writing we denote by $-\mathbf{\Pi}$ the partition $\langle -x_0, -x_1, ..., -x_n\rangle$.

(b) Bounding rational functions with real coefficients reduces to estimating polynomials since the inequality $|P/Q| < \varepsilon$ with $Q > 0$ is equivalent to the pair of inequalities $P - \varepsilon Q < 0$ and $P + \varepsilon Q > 0$. However, in our case the denominator is W which is very close to one, so an upper bound of the modulus of the numerator and a lower bound of W give an equivalent accuracy.

Proof. Use Note 4 (a) and the partition $\mathbf{\Pi} = -\langle \frac{17}{10}, \frac{27}{20}, \frac{4}{5}, \frac{3}{10}, \frac{3}{20}, 0\rangle$. ∎

To write the equation for $\delta = g - g_0$ in an integral form suitable for a contraction argument, we would need the fundamental solution of the linear operator on the left side of (124). Of course this cannot be done in closed form; once more, we use a pair of polynomials close enough in L^∞ to a fundamental system and estimate the errors introduced in this way. The pair of polynomials is obtained in the same way as g_0 was found, and is given by

$$J_1(t) = 1 - \frac{9489s^2}{2932} + \frac{1350s^3}{4721} + \frac{359s^4}{199} - \frac{1526s^5}{3719} - \frac{708s^6}{1633} + \frac{503s^7}{2201} - \frac{211s^8}{6486}\ (s = t - t_0) \tag{125}$$

$$J_2(t) = s - \frac{48s^2}{659797} - \frac{2941s^3}{2730} + \frac{675s^4}{4873} + \frac{1832s^5}{4745} - \frac{2305s^6}{19401} - \frac{677s^7}{14054} + \frac{1573s^8}{53783} - \frac{531s^9}{128216}\ (s = t - t_0)$$

Let $W = J_1 J_2' - J_2 J_1'$ be their Wronskian. The equation that the pair (J_1, J_2) satisfies is

$$f'' + Af' + Bf = 0 \tag{126}$$

where

$$A := \frac{J_2 J_1'' - J_1 J_2''}{W},\ B := \frac{J_2'' J_1' - J_1'' J_2'}{W} \tag{127}$$

The Green's function associated to (126) is

$$\mathcal{G}(s, t) = W(s)^{-1}[J_2(t)J_1(s) - J_1(t)J_2(s)] \tag{128}$$

We define the linear operators K_1, K_2 such that

$$K_1[f](t) = \int_{t_0}^{t} \mathcal{G}(s, t) f(s) ds,\ K_2[f](t) = \int_{t_0}^{t} \frac{\partial \mathcal{G}(s, t)}{\partial t} f(s) ds, \tag{129}$$

We can now rewrite now (124) as an equivalent integral system.

Let $B_1 = 12g_0 + B$, $\boldsymbol{\delta} = (\delta, \delta')$ and $r(\boldsymbol{\delta}(s), s) = A\delta' + B_1\delta + 6\delta^2$, $\mathbf{J}_1 = (J_1, J_1')$, $\mathbf{J}_2 = (J_2, J_2')$ and vector operator $\mathbf{K}$ defined by $\mathbf{K}[f] = (K_1[f], K_2[f])$. We have

$$\boldsymbol{\delta} = a_1 \mathbf{J}_1 + a_2 \mathbf{J}_2 - \mathbf{K}R + \mathbf{K}r = \hat{N}\boldsymbol{\delta} \tag{130}$$

and (18) implies

$$|a_1| < \alpha_1 := 1/290 \text{ and } |a_2| < \alpha_2 := 1/152 \tag{131}$$

To analyze the integral system we first estimate the various quantities in (130).

Proposition 17. *The following estimates hold in the sup norm on* $[t_0, 0]$

$$\max\{\|J_1\|, \|J_1/W\|\} \leqslant \frac{6}{5};\ \max\{\|J_2\|, \|J_2/W\|\} \leqslant \frac{3}{7},\ \|J_1'\| \leqslant \frac{5}{2},\ \|J_2'\| \leqslant \frac{21}{20} \tag{132}$$

$$|W-1| < 1/500,\ \|A\| < 1/1216,\ \|B_1\| < 1/492; \tag{133}$$

$$\sup_{|a_1|<\alpha_1, |a_2|<\alpha_2} \|a_1 J_1 + a_2 J_2\| < 1/180; \qquad \sup_{|a_1|<\alpha_1, |a_2|<\alpha_2} \|a_1 J_1' + a_2 J_2'\| < 1/90 \tag{134}$$

Proof. The proofs are based on Note 4 (a) to estimate the polynomials and rational functions.

We illustrate the calculation on a rational function, B_1 on a sample interval, say $(-\frac{3}{2}, -\frac{11}{10})$, see below. We thus take $s = -13/10 + u/5$ and simplify the resulting expression. Denoting by E_j polynomials with ℓ^1 norm less than $1/1000$ we simply get on this interval

$$B_1 = \left(\frac{3}{2332} - \frac{2u}{60137} - \frac{22u^2}{3361} + \frac{6u^3}{7241} + E_1\right)/(1+E_2)$$

The derivative of the cubic has one root for $u = -[1,1]$; the value of the absolute value cubic there is $< 1.3 \cdot 10^{-2}$. Checking it at the endpoints of the interval as well, we see that this is its maximum absolute value. The other calculations are as straightforward as this, so we naturally omit the details.

We found it convenient to use a different partition $\mathbf{\Pi}$ for estimating maximal absolute values of each function. We chose partitions $-\langle \frac{17}{10}, \frac{1}{2}, 0\rangle$ and $-\langle \frac{17}{10}, \frac{11}{10}, \frac{7}{10}, 0\rangle$ for J_1 and J_1' respectively (*cf.* again Note 4, (b)); $-\langle \frac{17}{10}, \frac{11}{10}, \frac{1}{2}, 0\rangle$, $-\langle \frac{17}{10}, \frac{11}{10}, 0\rangle$ and $-\langle \frac{17}{10}, \frac{17}{20}, 0\rangle$ for J_2, J_2' and W respectively. Finally, we use Note 4 (b) to estimate A and B_1 using the partitions $-\langle \frac{17}{10}, \frac{13}{10}, 1, \frac{7}{10}, \frac{3}{10}, \frac{1}{10}, 0\rangle$ and $-\langle \frac{17}{10}, \frac{3}{2}, \frac{11}{10}, \frac{7}{10}, \frac{2}{5}, \frac{1}{4}, \frac{1}{10}, 0\rangle$ respectively.

For (134) we note that $f(s, \mathbf{a}) := a_1 J_1(s) + a_2 J_2(s)$ and $f'(s; \mathbf{a})$ are linear in (a_1, a_2) and thus for any s, the extrema of $|f(s;\mathbf{a})|$, $|f'(s, \mathbf{a})|$ are reached on the boundary. Thus we only need bounds when $a_1 = \alpha_1$, $a_2 = \pm\alpha_2$. The partition points chosen are $= -\langle \frac{17}{10}, \frac{1}{2}, 0\rangle$ for $f(s, \boldsymbol{\alpha})$, $-\langle \frac{17}{10}, \frac{4}{5}, 0\rangle$ for $f(s; \alpha_1, -\alpha_2)$, $-\langle \frac{17}{10}, \frac{9}{10}, 0\rangle$ for $f'(s; \boldsymbol{\alpha})$ and $-\langle \frac{17}{10}, \frac{7}{5}, \frac{11}{10}, 0\rangle$ for $f'(s; \alpha_1, -\alpha_2)$). ∎

To analyze (130) we use the norm $\|\boldsymbol{\delta}\|^{(\frac{1}{2})} = \max\{\|\delta\|, \frac{1}{2}\|\delta'\|\}$ where $\|f\|$ is the sup norm on $[t_0, 0]$.

Proposition 18. *(i) The nonlinear operator* $\hat{N}$ *in* (130) *is contractive in the disc* $\{\boldsymbol{\delta} : \|\boldsymbol{\delta}\|^{(\frac{1}{2})} \leqslant 1/158\}$*; the contractivity factor is* $< 1/6$.

(ii) We have $\varepsilon_1 = \|\delta - a_1 J_1 - a_2 J_2\| < 1/1500$, $\varepsilon_2 := \|\delta' - a_1 J_1' - a_2 J_2'\| < 1/658$.

(iii) Let $a = 87/469$ *and* $b = 41/134$. *We have*

$$|g(0) + a| < 1/167;\ |g'(0) - b| < \varepsilon = 1/108 \tag{135}$$

The proof of (i) is a simple calculation based on the estimates in Proposition 5. The integrals are estimated crudely, by placing an absolute value on all terms and multiplying with the length of the interval, 17/10.

For (ii) we note that

$$\varepsilon_1 \leqslant \|K_1\| \left(2\|A\|\|\boldsymbol{\delta}\|^{(\frac{1}{2})} + \|B_1\|\boldsymbol{\delta} + 6\|\boldsymbol{\delta}\|^2 + \|R\|\right)$$

(iii) At $t = 0$ $(s = 1.7)$ we have

$$|g(0) + a| \leqslant |g_0(0) + a| + \max_{|a_j|<\alpha_j, j=1,2} |a_1 J_1(0) + a_2 J_2(0)| + \varepsilon_1 < 1/167$$

where we used (ii), (123), (125) and (131); $g'(0)$ is estimated in a very similar way.

6. The Maclaurin series of y_t

Proposition 19. *The Maclaurin series of the function $t \mapsto g(t)$ converges in a disk of radius at least $R_0 = 1.85$.*

Proof. We define $c_i, i = 0, 1, 2, ...$ to be the Taylor coefficients of g at $t = 0$ and note that

$$c_2 = 3c_0^2 > 0 \tag{136}$$

The recurrence relation for the Taylor coefficients of the solution of (121) with initial condition $g(0) = c_0 = -a, g'(0) = c_1 = b$ is

$$(k+1)(k+2)c_{k+2} = 6\sum_{j=0}^{k} c_j c_{k-j};\ \ k > 1 \tag{137}$$

We now take the full range of initial conditions compatible with the error range (135) (for simplicity we take the larger of the two bounds, ε): $\tilde{c}_0 = -a + \varepsilon\sigma_1,\ \tilde{c}_1 = b + \varepsilon\sigma_2$ where $\sigma_1, \sigma_2 \in I = [-1, 1]$, and calculate the formal series solution at zero for these initial data; we denote by $\tilde{c}_i$ the Taylor coefficients thus calculated; we have

$$\tilde{c}_0 = -a + \varepsilon\sigma_1,\ \tilde{c}_1 = b + \varepsilon\sigma_2, \tilde{c}_2 = 3(-a + \varepsilon\sigma_1)^2,\ \tilde{c}_3 = 2(-a + \varepsilon\sigma_1)(b + \varepsilon\sigma_2) + \tfrac{1}{6} \tag{138}$$

The coefficients $-\tilde{c}_0, \tilde{c}_1, c_2, \tilde{c}_3$ in (138) can clearly be maximized/minimized by elementary means as functions of $(\sigma_1, \sigma_2) \in I^2$. The result is

$$0 < -\tilde{c}_0 < 1/5, 0 < \tilde{c}_1 < 6/19, 0 < \tilde{c}_2 < 1/8,\ 0 < \tilde{c}_3 < 1/15 \tag{139}$$

We write

$$(k+2)(k+1)|\tilde{c}_{k+2}| < 6\sum_{j=0}^{k} |\tilde{c}_j||\tilde{c}_{k-j}|;\ \ k \geqslant 2 \tag{140}$$

where $\tilde{c}_i, i = 0, ..., 3$ are taken to be the upper bounds in (139). We check that for $k = 0, 1, 2, 3$ we have

$$|\tilde{c}_k| < (k+1)/R_0^{k+2};\quad \text{where } R_0 = \frac{37}{20} = 1.85 \tag{141}$$

Lemma 20. *The inequality* (141) *is satisfied for all $k \geqslant 0$.*

Proof. We note that for any $\rho > 0$ the sequence $a_k = (k+1)\rho^{k+2}, k = 0, 1, 2, ...$ is an exact solution of the recurrence

$$(k+1)(k+2)a_{k+2} = 6\sum_{j=0}^{k} a_j a_{k-j}, \forall\ k \geqslant 0 \tag{142}$$

The rest is straightforward induction. ∎

Proposition 19 now follows from (141). ∎

7. End of proof of Theorem 1

We have already shown that in each domain Ω_I, $\Omega_1 \cup \Omega_2$ the unique solution we obtained equals the tritronquée $h_t(x)$. After changes of variables, by matching at $z = z_0 = 1.7e^{i\pi/5}$(*i.e.,* at $t = t_0 = -1.7$), we then proved that $y_t(z)$ has a convergent Maclaurin series with radius of convergence $\geqslant 1.85$. Therefore, it follows that $y_t(z)$ is pole-free in the domain $\left\{-\frac{3}{5\pi} \leqslant \arg z \leqslant \frac{\pi}{5}, |z| \geqslant 1.7\right\} \cup \{z : |z| \leqslant 1.85\}$. By the symmetry of the solution, see Remark 3 and the Schwartz reflection principle, regularity also follows for $\arg z \in [\pi/5, \pi]$.

In the process, we determined $y_t(0)$ and $y_t'(0)$ to within $< 10^{-2}$ rigorous error bounds (see (135) and (121)) in agreement with [8].

8. Appendix: The concrete expressions of various terms

$$r_5(\zeta) = -\frac{53}{64}\frac{S^2}{\zeta^2} + \frac{161}{1728}\frac{S^4}{\zeta^4} - \frac{35}{41472}\frac{S^6}{\zeta^6},\ r_6(\zeta) = -\frac{995}{2304}\frac{S^3}{\zeta^3} + \frac{301}{20736}\frac{S^5}{\zeta^5} - \frac{11}{124416}\frac{S^7}{\zeta^7} \tag{143}$$

$$r_7(\zeta) = -\frac{392}{625} - \frac{5551}{9216}\frac{S^2}{\zeta^2} - \frac{1417}{165888}\frac{S^4}{\zeta^4} + \frac{289}{248832}\frac{S^6}{\zeta^6} - \frac{23}{2985984}\frac{S^8}{\zeta^8} \tag{144}$$

$$r_8(\zeta) = \frac{225S}{512\zeta} - \frac{2051}{9216}\frac{S^3}{\zeta^3} - \frac{241}{55296}\frac{S^5}{\zeta^5} + \frac{23}{186624}\frac{S^7}{\zeta^7} - \frac{5}{8957952}\frac{S^9}{\zeta^9} \tag{145}$$

$$r_9(\zeta) = -\frac{81}{32768}\frac{S^2}{\zeta^2} + \frac{43}{16384}\frac{S^4}{\zeta^4} - \frac{947}{1327104}\frac{S^6}{\zeta^6} + \frac{215}{23887872}\frac{S^8}{\zeta^8} - \frac{25}{859963392}\frac{S^{10}}{\zeta^{10}} \tag{146}$$

$$q_5(\zeta) = -\frac{539}{384}\frac{S}{\zeta^2} + \frac{307}{864}\frac{S^3}{\zeta^4} - \frac{35}{6912}\frac{S^5}{\zeta^6},\ q_6(\zeta) = -\frac{1361}{1152}\frac{S^2}{\zeta^3} + \frac{727}{10368}\frac{S^4}{\zeta^5} - \frac{77}{124416}\frac{S^6}{\zeta^7} \tag{147}$$

$$q_7(\zeta) = -\frac{95}{96}\frac{S}{\zeta^2} - \frac{3817}{165888}\frac{S^3}{\zeta^4} + \frac{277}{41472}\frac{S^5}{\zeta^6} - \frac{23}{373248}\frac{S^7}{\zeta^8} \tag{148}$$

$$q_8(\zeta) = -\frac{621S^2}{1024\zeta^3} - \frac{1591}{82944}\frac{S^4}{\zeta^5} + \frac{623}{746496}\frac{S^6}{\zeta^7} - \frac{5}{995328}\frac{S^8}{\zeta^9} \tag{149}$$

$$q_9(\zeta) = \frac{15}{2048}\frac{S^3}{\zeta^4} - \frac{3515}{884736}\frac{S^5}{\zeta^6} + \frac{1675}{23887872}\frac{S^7}{\zeta^8} - \frac{125}{429981696}\frac{S^9}{\zeta^{10}} \tag{150}$$

$$E_5(\zeta) = -\frac{269}{576}\frac{S^3}{\zeta} + \frac{61}{10368}\frac{S^5}{\zeta^3},\ E_6(\zeta) = -\frac{1691}{20736}\frac{S^4}{\zeta^2} + \frac{353}{497664}\frac{S^6}{\zeta^4} \tag{151}$$

$$E_7(\zeta) = \frac{95}{576}\frac{S^3}{\zeta} - \frac{1915}{248832}\frac{S^5}{\zeta^3} + \frac{25}{373248}\frac{S^7}{\zeta^5},\ E_8(\zeta) = \frac{25}{768}\frac{S^4}{\zeta^2} - \frac{125}{165888}\frac{S^6}{\zeta^4} + \frac{625}{143327232}\frac{S^8}{\zeta^6} \tag{152}$$

$$\begin{aligned} E_M(\rho) = &\left(\frac{269|S|^3}{288} + \frac{61|S|^5}{15552}\right)\rho^{-3/2} + \left(\frac{1691|S|^4}{20736} + \frac{353|S|^6}{995328}\right)\rho^{-2} \\ &+ \left(\frac{95|S|^3}{288} + \frac{1915|S|^5}{373248} + \frac{5|S|^7}{186624}\right)\rho^{-5/2} + \left(\frac{25|S|^4}{768} + \frac{125|S|^6}{331776} + \frac{625|S|^8}{429981696}\right)\rho^{-3} \end{aligned} \tag{153}$$

$$M_q = \sum_{j=3}^{7} M_{q,j}\rho^{-j/2}, \tag{154}$$

where

$$M_{q,3} = \frac{539}{1536}|S| + \frac{307}{3456}|S|^3 + \frac{35}{27648}|S|^5,\ M_{q,4} = \frac{1361}{5184}|S|^2 + \frac{727}{46656}|S|^4 + \frac{77}{559872}|S|^6 \tag{155}$$

$$M_{q,5} = \frac{19}{96}|S| + \frac{3817}{829440}|S|^3 + \frac{277}{207360}|S|^5 + \frac{23}{1866240}|S|^7 \tag{156}$$

$$M_{q,6} = \frac{621}{5632}|S|^2 + \frac{1591}{456192}|S|^4 + \frac{623}{4105728}|S|^6 + \frac{5}{5474304}|S|^8 \tag{157}$$

$$M_{q,7} = \frac{5}{4096}|S|^3 + \frac{3515}{5308416}|S|^5 + \frac{1675}{143327232}|S|^7 + \frac{125}{2579890176}|S|^9 \tag{158}$$

$$M_{L,q} = \sum_{j=3}^{7} M_{L,q,j}\rho^{-j/2}, \tag{159}$$

where

$$M_{L,q,3} = \frac{77}{192}|S| + \frac{307}{3024}|S|^3 + \frac{5}{3456}|S|^5,\ M_{L,q,4} = \frac{1361}{4608}|S|^2 + \frac{727}{41472}|S|^4 + \frac{77}{497664}|S|^6 \tag{160}$$

$$M_{L,q,5} = \frac{95}{432}|S| + \frac{3817}{746496}|S|^3 + \frac{277}{186624}|S|^5 + \frac{23}{1679616}|S|^7 \tag{161}$$

$$M_{L,q,6} = \frac{621}{5120}\,|S|^2 + \frac{1591}{414720}\,|S|^4 + \frac{623}{3732480}\,|S|^6 + \frac{1}{995328}\,|S|^8 \tag{162}$$

$$M_{L,q,7} = \frac{15}{11264}\,|S|^3 + \frac{3515}{4866048}\,|S|^5 + \frac{1675}{131383296}\,|S|^7 + \frac{125}{2364899328}\,|S|^9 \tag{163}$$

$$M_{G,1} = \sum_{j=0}^{2} m_{j,1}\rho^{-j/2} \tag{164}$$

$$m_{0,1} = \frac{784}{3125}\left(1+\sqrt{2}\right) + \frac{5551}{11520}\,|S|^2 + \frac{23}{3732480}\,|S|^8 + \frac{289}{311040}\,|S|^6 + \frac{1417}{207360}\,|S|^4 \tag{165}$$

$$m_{1,1} = \frac{75}{256}\,|S| + \frac{2051}{13824}\,|S|^3 + \frac{241}{82944}\,|S|^5 + \frac{5}{13436928}\,|S|^9 + \frac{23}{279936}\,|S|^7 \tag{166}$$

$$m_{2,1} = \frac{947}{2322432}\,|S|^6 + \frac{25}{1504935936}\,|S|^{10} + \frac{215}{41803776}\,|S|^8 + \frac{43}{28672}\,|S|^4 + \frac{81}{57344}\,|S|^2 \tag{167}$$

$$M_{G,2} = \frac{161|S|^4}{4320} + \frac{53|S|^2}{160} + \frac{7|S|^6}{20736} + \left(\frac{995|S|^3}{6912} + \frac{11|S|^7}{373248} + \frac{301|S|^5}{62208}\right)\rho^{-1/2}, \tag{168}$$

$$M_{G,3} \;=\; \frac{161}{1620}\,|S|^4 \;+\; \frac{7}{7776}\,|S|^6 \;+\; \frac{53}{60}\,|S|^2 \;+\; \left(\frac{4975}{13824}\,|S|^3 + \frac{1505}{124416}\,|S|^5 + \frac{55}{746496}\,|S|^7\right)\rho^{-1/2} \tag{169}$$

$$t_5(\zeta) = \frac{161}{1728}\,\frac{S^4}{\zeta^5} - \frac{53}{64}\,\frac{S^2}{\zeta^3} - \frac{35}{41472}\,\frac{S^6}{\zeta^7},\ t_6(\zeta) = -\frac{23}{62208}\,\frac{S^7}{\zeta^8} + \frac{35}{768}\,\frac{S^5}{\zeta^6} - \frac{1631}{2304}\,\frac{S^3}{\zeta^4} \tag{170}$$

$$t_7(\zeta) = -\frac{11}{373248}\,\frac{S^8}{\zeta^9} + \frac{301}{62208}\,\frac{S^6}{\zeta^7} - \frac{995}{6912}\,\frac{S^4}{\zeta^5} \tag{171}$$

$$u_5(\zeta) = \frac{161}{3456}\,\frac{S^4}{\zeta^3} - \frac{53}{128}\,\frac{S^2}{\zeta} - \frac{35}{82944}\,\frac{S^6}{\zeta^5},\ u_6(\zeta) = \frac{47}{124416}\,\frac{S^7}{\zeta^6} - \frac{1631}{41472}\,\frac{S^5}{\zeta^4} + \frac{913}{4608}\,\frac{S^3}{\zeta^2} \tag{172}$$

$$u_7(\zeta) = \frac{173}{746496}\,\frac{S^8}{\zeta^7} - \frac{3479}{124416}\,\frac{S^6}{\zeta^5} + \frac{1843}{4608}\,\frac{S^4}{\zeta^3},\ u_8(\zeta) = \frac{11}{559872}\,\frac{S^9}{\zeta^8} - \frac{301}{93312}\,\frac{S^7}{\zeta^6} + \frac{995}{10368}\,\frac{S^5}{\zeta^4} \tag{173}$$

$$\tau_5(\zeta) = -\frac{161S^4}{8640\zeta^5} + \frac{53S^2}{192\zeta^3} + \frac{5S^6}{41472\zeta^7},\ \tau_6(\zeta) = \frac{23S^7}{497664\zeta^8} - \frac{35S^5}{4608\zeta^6} + \frac{1631S^3}{9216\zeta^4} \tag{174}$$

$$\tau_7(\zeta) = \frac{11S^8}{3359232\zeta^9} - \frac{43S^6}{62208\zeta^7} + \frac{199S^4}{6912\zeta^5} \tag{175}$$

$$\tilde{t}_7(\zeta) = -\frac{161}{3456}\,\frac{S^4}{\zeta^5} + \frac{265}{384}\,\frac{S^2}{\zeta^3} + \frac{25}{82944}\,\frac{S^6}{\zeta^7},\ \tilde{t}_8(\zeta) = \frac{23}{165888}\,\frac{S^7}{\zeta^8} - \frac{35}{1536}\,\frac{S^5}{\zeta^6} + \frac{1631}{3072}\,\frac{S^3}{\zeta^4} \tag{176}$$

$$\tilde{t}_9(\zeta) = \frac{77}{6718464}\,\frac{S^8}{\zeta^9} - \frac{301}{124416}\,\frac{S^6}{\zeta^7} + \frac{1393}{13824}\,\frac{S^4}{\zeta^5} \tag{177}$$

$$\nu_5(\zeta) = -\frac{161}{10368}\,\frac{S^4}{\zeta^3} + \frac{53}{128}\,\frac{S^2}{\zeta} + \frac{7}{82944}\,\frac{S^6}{\zeta^5},\ \nu_6(\zeta) = -\frac{47}{746496}\,\frac{S^7}{\zeta^6} + \frac{1631}{165888}\,\frac{S^5}{\zeta^4} - \frac{913}{9216}\,\frac{S^3}{\zeta^2} \tag{178}$$

$$\nu_7(\zeta) = -\frac{173}{5225472}\,\frac{S^8}{\zeta^7} + \frac{3479}{622080}\,\frac{S^6}{\zeta^5} - \frac{1843}{13824}\,\frac{S^4}{\zeta^3},\ \nu_8(\zeta) = -\frac{11}{4478976}\,\frac{S^9}{\zeta^8} + \frac{301}{559872}\,\frac{S^7}{\zeta^6} - \frac{995}{41472}\,\frac{S^5}{\zeta^4} \tag{179}$$

$$\tilde{u}_7(\zeta) = -\frac{805}{20736}\,\frac{S^4}{\zeta^3} + \frac{265}{256}\,\frac{S^2}{\zeta} + \frac{35}{165888}\,\frac{S^6}{\zeta^5},\ \tilde{u}_8(\zeta) = -\frac{47}{248832}\,\frac{S^7}{\zeta^6} + \frac{1631}{55296}\,\frac{S^5}{\zeta^4} - \frac{913}{3072}\,\frac{S^3}{\zeta^2} \tag{180}$$

$$\tilde{u}_9(\zeta) = -\frac{173}{1492992}\,\frac{S^8}{\zeta^7} + \frac{24353}{1244160}\,\frac{S^6}{\zeta^5} - \frac{12901}{27648}\,\frac{S^4}{\zeta^3},\ \tilde{u}_{10}(\zeta) = -\frac{11}{1119744}\,\frac{S^9}{\zeta^8} + \frac{301}{139968}\,\frac{S^7}{\zeta^6} - \frac{995}{10368}\,\frac{S^5}{\zeta^4} \tag{181}$$

$$p_5(\zeta) = -\frac{161}{25920}\,\frac{S^4}{\zeta^3} + \frac{53}{192}\,\frac{S^2}{\zeta} + \frac{1}{41472}\,\frac{S^6}{\zeta^5}, p_6(\zeta) = \frac{1001}{829440}\,\frac{S^5}{\zeta^4} - \frac{65}{18432}\,\frac{S^3}{\zeta^2} - \frac{17}{2985984}\,\frac{S^7}{\zeta^6} \tag{182}$$

$$p_7(\zeta) = \frac{17}{19440}\,\frac{S^6}{\zeta^5} - \frac{185}{47029248}\,\frac{S^8}{\zeta^7} - \frac{137}{4608}\,\frac{S^4}{\zeta^3}, p_8(\zeta) = -\frac{199}{41472}\,\frac{S^5}{\zeta^4} + \frac{43}{559872}\,\frac{S^7}{\zeta^6} - \frac{11}{40310784}\,\frac{S^9}{\zeta^8} \tag{183}$$

(184)
$$M_{G,4,0} := \frac{161}{25920}\,|S|^4 + \frac{53}{192}\,|S|^2 + \frac{1}{41472}\,|S|^6 + \left(\frac{17}{2985984}\,|S|^7 + \frac{1001}{829440}\,|S|^5 + \frac{65}{18432}\,|S|^3\right)\rho^{-1/2}$$
$$+\left(\frac{185}{47029248}\,|S|^8 + \frac{137}{4608}\,|S|^4 + \frac{17}{19440}\,|S|^6\right)\rho^{-1} + \left(\frac{43}{559872}\,|S|^7 + \frac{11}{40310784}\,|S|^9 + \frac{199}{41472}\,|S|^5\right)\rho^{-3/2}$$

$$M_{G,4,1} = \frac{|S|^6}{6912} + \frac{161|S|^4}{6480} + \frac{53|S|^2}{96} + \left(\frac{2401|S|^5}{92160} + \frac{761|S|^3}{2048} + \frac{497|S|^7}{2985984}\right)\rho^{-1/2}$$
$$+\left(\frac{1711|S|^6}{155520} + \frac{3083|S|^8}{47029248} + \frac{3673|S|^4}{13824}\right)\rho^{-1} + \left(\frac{199|S|^5}{4608} + \frac{559|S|^7}{559872} + \frac{187|S|^9}{40310784}\right)\rho^{-3/2} \tag{185}$$

$$M_{G,5} = \frac{1417}{331776}\,|S|^4 + \frac{23}{5971968}\,|S|^8 + \frac{289}{497664}\,|S|^6 + \frac{5551}{18432}\,|S|^2 + \frac{196}{625}$$
$$+\left(\frac{241}{138240}\,|S|^5 + \frac{2051}{23040}\,|S|^3 + \frac{45}{256}\,|S| + \frac{1}{4478976}\,|S|^9 + \frac{23}{466560}\,|S|^7\right)\rho^{-1/2}$$
$$+\left(\frac{25}{2579890176}\,|S|^{10} + \frac{215}{71663616}\,|S|^8 + \frac{947}{3981312}\,|S|^6 + \frac{43}{49152}\,|S|^4 + \frac{27}{32768}\,|S|^2\right)\rho^{-1} \tag{186}$$

$$M_{G,6} = \frac{161}{3456}\,|S|^4 + \frac{35}{82944}\,|S|^6 + \frac{53}{128}\,|S|^2 + \left(\frac{199}{1152}\,|S|^3 + \frac{301}{51840}\,|S|^5 + \frac{11}{311040}\,|S|^7\right)\rho^{-1/2} \tag{187}$$

$$M_{G,7} = \frac{1771}{57600}\,|S|^4 + \frac{11}{55296}\,|S|^6 + \frac{583}{1280}\,|S|^2 + \left(\frac{37513}{138240}\,|S|^3 + \frac{161}{13824}\,|S|^5 + \frac{529}{7464960}\,|S|^7\right)\rho^{-1/2}$$
$$+\left(\frac{12139}{290304}\,|S|^4 + \frac{2623}{2612736}\,|S|^6 + \frac{671}{141087744}\,|S|^8\right)\rho^{-1} \tag{188}$$

8.1. **Values of intermediate constants for $\rho = 3$.** The numerical values of these constants might be helpful to the reader who would like to double-check the estimates. These are:

$$J_M = 0.282580\cdots, j_m = 0.64374\cdots, Y_{1,M} = 1.16314\cdots, Y_{1,R,M} = 0.132618\cdots, E_M = 0.0490292\cdots$$
$$z_{2,R,M} = 0.54226\cdots, z_{2,M} = 0.91863\cdots, M_q = 0.066702\cdots, M_{L,q} = 0.075708\cdots, V_M = 0.2239\cdots, T_M = 0.0385\cdots$$
$$M_1 = 1.13838\cdots, M_2 = 0.04303\cdots, M_3 = 0.28346\cdots, M_4 = 0.45227\cdots, M_5 = 0.05430\cdots, M_6 = 0.00231\cdots, M_7 = 0.02018\cdots$$

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